Math 61 Midterm 1

TOTAL POINTS

13 / 13

QUESTION 1

1 Induction Question **4 / 4**

✓ **- 0 pts** **Correct**

- 1.5 pts Base case was trivialized

- 0.75 pts Base case and/or induction step argument does not explain both inclusions between a pair of sets.

- 1 pts Induction step carried out incorrectly in form (obscuring the role of induction in the proof)

- 1 pts Induction step carried out incorrectly in content

- 0.5 pts Misunderstanding of union

- 0.5 pts Unclear logic in base case

- 0.5 pts Handles arbitrary elements or sets incorrectly

- 0.5 pts Misunderstanding or unclear use of equality and/or implication

- 0.5 pts Misunderstanding of set builder notation or sets and their cardinalities

- 0.25 pts Minor unpacking error

- 0.25 pts Misuse of notation

- 1 pts Misunderstanding of cartesian product

QUESTION 2

2 Relation Question **4 / 4**

Part (a): (i) $$R$$ can be both anti-symmetric and symmetric simultaneously, or (ii) $$R$$ can be not(anti-symmetric) and not(symmetric) simultaneously.

✓ **- 0 pts** **Correct example: gave a relation which was both symmetric and anti-symmetric, or a relation which was neither.**

- 1 pts Unclear or imprecise mathematical statements made. For example, the argument did not give a clear explanation of both properties, or gave some correct examples (of a relation being both or neither properties), but also some incorrect examples.

- 2 pts Missing or incorrect example, or major misunderstandings

Part (b): $$ R $$ can be not(anti-reflexive) and not(reflexive) simultaneously

✓ **- 0 pts** **Correct: gave an example of a relation which was neither reflexive nor anti-reflexive. (Or, gave the example of $$ X = \emptyset $$ and $$ R = \emptyset $$)**

- 1 pts Unclear or imprecise mathematical statements made. For example, the argument did not give a clear explanation of both properties, or gave some correct examples, but also some incorrect examples.

- 2 pts Missing or incorrect example, or major misunderstandings

QUESTION 3

3 Function Question **3 / 3**

✓ **- 0 pts** **Correct**

- 1 pts incomplete or incorrect argument for injectivity when n = 1

- 1 pts incomplete or incorrect argument for surjectivity

- 1 pts incomplete or incorrect argument for non-injectivity for n >1

QUESTION 4

4 Counting Question **2 / 2**

✓ **- 0 pts** **Correct**

- 0.5 pts 16! ways with Averie first and 16! ways with Charlie last

- 0.5 pts 15! ways with Averie first and Charlie last

- 1 pts 2(16!)-15! total by Inclusion-Exclusion Principle

Problem 1. Let $n \geq 2$ be a natural number. Let $A_1, \dots, A_n$ and C be arbitrary sets. Using mathematical induction, show that

$$\left(\bigcup_{i=1}^{n} A_i\right) \times C = \bigcup_{i=1}^{n} (A_i \times C).$$

Base case: when $n=2$

$LHS = \left(\bigcup_{i=1}^{2} A_i\right) \times C$ $\quad\quad RHS = \bigcup_{i=1}^{2} (A_i \times C) = (A_1 \times C) \cup (A_2 \times C)$

$= (A_1 \cup A_2) \times C$

To prove, we can use distributive property $(A_1 \cup A_2) \times C = (A_1 \times C) \cup (A_2 \times C)$:

① prove that for any $(a,c) \in (A_1 \cup A_2) \times C \Rightarrow (a,c) \in (A_1 \times C) \cup (A_2 \times C)$.
For any (a,c) s.t. $(a,c) \in (A_1 \cup A_2) \times C$, we know that $a \in A_1 \cup A_2$ and $c \in C$.
Hence, we have $a \in A_1$ or $a \in A_2$ and $c \in C$. Knowing this, we have $a \in A_1$ and $c \in C$ or $a \in A_2$ and $c \in C$.
So this gives us $(a,c) \in A_1 \times C$ or $(a,c) \in A_2 \times C$ which gives us $(a,c) \in (A_1 \times C) \cup (A_2 \times C)$ ✓.

② Prove that for any $(a,c) \in (A_1 \times C) \cup (A_2 \times C) \Rightarrow (a,c) \in (A_1 \cup A_2) \times C$.
For any (a,c) s.t. $(a,c) \in (A_1 \times C) \cup (A_2 \times C)$, we know that $(a,c) \in (A_1 \times C)$ or $(a,c) \in (A_2 \times C)$.
This gives us $a \in A_1$ and $c \in C$ or $a \in A_2$ and $c \in C$. This means that $a \in A_1$ or $a \in A_2$ and $c \in C$.
Hence we have $a \in A_1 \cup A_2$ and $c \in C$ so $(a,c) \in (A_1 \cup A_2) \times C$. ✓

Thus we have proved that $(A_1 \cup A_2) \times C = (A_1 \times C) \cup (A_2 \times C)$, so we can use this property.

Continuing our base case: (using distributive prop.)

$LHS = \left(\bigcup_{i=1}^{2} A_i\right) \times C = (A_1 \cup A_2) \times C = (A_1 \times C) \cup (A_2 \times C) = RHS$ ✓

Induction step: inductive hypothesis: Assume $k \in \mathbb{N}$ and $k \geq 2$ s.t. $\left(\bigcup_{i=1}^{k} A_i\right) \times C = \bigcup_{i=1}^{k} (A_i \times C)$.

We want to show that: $LHS = \left(\bigcup_{i=1}^{k+1} A_i\right) \times C = \bigcup_{i=1}^{k+1} (A_i \times C) = RHS$.

$$LHS = \left(\bigcup_{i=1}^{k+1} A_i\right) \times C = \left(\left(\bigcup_{i=1}^{k} A_i\right) \cup A_{k+1}\right) \times C = \underbrace{\left(\left(\bigcup_{i=1}^{k} A_i\right) \times C\right)}_{\text{we know this by our inductive hypothesis}} \cup (A_{k+1} \times C)$$

$$= \bigcup_{i=1}^{k} (A_i \times C) \cup (A_{k+1} \times C)$$

$$= \bigcup_{i=1}^{k+1} (A_i \times C) = RHS \checkmark$$

By induction, we have shown that $\left(\bigcup_{i=1}^{n} A_i\right) \times C = \bigcup_{i=1}^{n} (A_i \times C)$ for all $n \in \mathbb{N}$ and $n \geq 2$.

1 Induction Question 4 / 4

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Problem 2. Let R be a relation on a set X.

(a) Explain in words why the statement "R is anti-symmetric" is not the negation of the statement "R is symmetric". Provide examples to illustrate your explanation.

(b) Explain in words why the statement "R is anti-reflexive" is not the negation of the statement "R is reflexive". Provide examples to illustrate your explanation.

(a) "R is anti-symmetric" is NOT the negation of "R is symmetric" because when one statement is the negation of another, that would imply that if 1 statement is true, the other must be false. We know that "R is anti-symmetric" is defined by: for all $x, y \in S$, if xRy and yRx, then $x=y$. We also know that "R is symmetric" means for all $x, y \in S$, if xRy then yRx. We see that these two statements can simultaneously both be true since the "if" conditional statement of anti-symmetry includes both the "if" and "then" statements of symmetry.

For a clearer understanding, let's look at an example:

We let $R = \{(1,1)\}$.

To prove anti-symmetry: xRy is true b/c $1R1$, and yRx is also true b/c $1R1$ so then $x=y$ which is indeed true b/c $1=1$.
$\therefore$ R is antisymmetric.

To prove symmetry: The if condition xRy is true b/c $1R1$. Then yRx is also true b/c $1R1$. $\therefore$ R is symmetric.

R is both antisymmetric and symmetric so they are NOT negations of one another.

(b) "R is anti-reflexive" is NOT the negation of "R is reflexive because as established in (a), if 1 statement is the negation of another, they cannot both simultaneously be false (since if 1 is false, the other must be true).
We know that "R is anti-reflexive" is defined for all $x \in S$, x is NEVER related to itself. We know that "R is reflexive" is defined that for ALL $x \in S$, xRx.
These can simultaneously both be false.
To gain a clearer understanding, let's look at an example:

(see next page) →

let $S=\{1,2,3,4,5\}$

and let $R=\{(1,1)\}$

To prove it is NOT anti-reflexive:

for all $x \in S$, x must never be related to itself but in relation R we see $1R1$, i.e. xRx $\therefore$ R is NOT antireflexive.

To prove it is NOT reflexive:

For R to be reflexive, xRx for ALL $x \in S$. We see that when $x=1$, $x \in S$ and xRx HOWEVER when $x=2$, $x \in S$ but x is NOT related to x so it is missing $(2,2)$. Similarly, R is missing $(3,3)$, $(4,4)$, and $(5,5)$ which are all needed for R to be reflexive.

$\therefore$ R is NOT reflexive.

We have proved that R is not anti-reflexive and R is not reflexive. They are simultaneously both false so antireflexive and reflexive must not be negations of one another.

2 Relation Question 4 / 4

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✓ - 0 pts **Correct example: gave a relation which was both symmetric and anti-symmetric, or a relation which was neither.**

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Part (b): $$ R $$ can be not(anti-reflexive) and not(reflexive) simultaneously

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- **1 pts** Unclear or imprecise mathematical statements made. For example, the argument did not give a clear explanation of both properties, or gave some correct examples, but also some incorrect examples.

- **2 pts** Missing or incorrect example, or major misunderstandings

Problem 3. Let n be a positive natural number. Let $X = \{i \in \mathbb{N} : 1 \le i \le n\}$. Denote by $\mathcal{P}(X)$ the power set of X, and let $\mathcal{P}^*(X) := \mathcal{P}(X) \setminus \{\emptyset\}$ denote the set of subsets of X that are not empty. Consider the function

all subsets of X

$$f : \mathcal{P}^*(X) \to X$$

which sends each non-empty subset of X to its least element. For instance, $f(\{1,3\}) = 1$. For which values of n is f injective, surjective, or bijective? Carefully motivate your arguments.

① For f to be injective, that means for every $y \in ran(f)$, there must be a unique $x \in X$ where $f(x) = y$. Another way to state this would be for every least element of a set, denoted by y, there must be a unique subset $S \in P^*(X)$ where $f(S) = y$.

let value $n=1$, then $X = \{1\}$ and $P^*(x) = \{\{1\}\}$ so when $f(S) = 1$, $S = \{1\}$ would be the only unique $S \in P^*(x)$ where $f(S) = 1$ because $S = \{1\}$ is the only set in the powerset of X. If $n > 1$, however, and the least element, y, is equal to 1, we know that $S = \{1\}$ is a potential solution, but so is $S = \{n, n-1, \dots, 1\}$, therefore, for any $n > 1$, f will no longer be injective since multiple sets in $P^*(x)$ will contain 1.

$\boxed{\text{So } f \text{ is injective only when } n=1.}$

② If f is surjective, then for every $y \in Y$, there is an $x \in X$ s.t. $f(x) = y$. In other words, for every least element, denoted by y, there is a set $S \in P^*(x)$ s.t. $f(S) = y$. By this definition, f is surjective for all $n \in \mathbb{N}$. To show this, we know that for $f: P^*(x) \to X$, the range of f is all natural numbers less than or equal to n. We also know that for any set $X = \{1, 2, \dots n\}$, The subsets $\{n\}, \{n-1\}, \dots, \{1\}$ are all in $P^*(x)$. Knowing this, we have $f(\{n\}) = n$, $f(\{n-1\}) = n-1, \dots, f(\{1\}) = 1$. Thus it is evident that the least element outputted will hit every value on the range of f.

$\boxed{\text{so } f \text{ is surjective for all } n \in \mathbb{N}}$

③ f is bijective when f is both injective and surjective, so we find the intersection of these 2 n values: $1 \cap \mathbb{N}$ and find that $\boxed{f \text{ is bijective only when } n=1.}$

3 Function Question 3 / 3

✓ **- 0 pts** **Correct**

- 1 pts incomplete or incorrect argument for injectivity when n = 1

- 1 pts incomplete or incorrect argument for surjectivity

- 1 pts incomplete or incorrect argument for non-injectivity for n >1

Problem 4. A teacher wants to arrange their 17 students in a single line. There are two students Averie and Charlie, in this class. How many ways are there for the students to line up so that either Averie is first in line or Charlie is last (or both)?

A C

16 other students

OR

. . . . C

OR

A C

15 other students

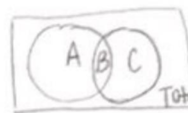
The # of ways to line up w/ Averie 1st is 16!

The # of ways to line up w/ Charlie last is also 16!

The # of ways to line up w/ Averie 1st <u>and</u> Charlie last is 15!

let A be the set of total ways Averie could be 1st

let C be the set of total ways for Charlie to be last



Total ways to line up

$|A| + |C| - |A \cap C| = |A \cup C|$, by <u>inclusion-exclusion</u> (we must use inclusion exclusion to avoid double counting)

$|A| = 16!$

$|C| = 16!$

$|A \cap C| = 15!$

$\Rightarrow$ So $|A \cup C| = 16! + 16! - 15!$

$$|A \cup C| = 2(16!) - 15!$$

So there are $2(16!) - 15!$ ways for the students to line up so that either Charlie is last or Averie is first or both.

4 Counting Question 2 / 2

✓ **- 0 pts** **Correct**

- 0.5 pts 16! ways with Averie first and 16! ways with Charlie last

- 0.5 pts 15! ways with Averie first and Charlie last

- 1 pts 2(16!)-15! total by Inclusion-Exclusion Principle