Name:

Student ID#:

MATH 61 - MIDTERM EXAM 1

0.1. **Instructions.** This is a 50 minute exam. You should feel free to quote any theorems proved in class, as well as anything proved in the homework or discussion section. There are 7 questions—on the real exam, you are required to do the first true/false question, and choose 5 of the remaining 6. Only 5 problems other than the true/false question will be graded so *you should indicate which problems you want graded*, in the case that you attempt all 6. Each question is worth 10 points. Unless otherwise specified, you are required to justify your answers.

Exercise 0.1. Indicate whether the following statements are true or false. You do not need to justify your answer.

(1) If $f : X \to Y$ and $g : Y \to Z$ are one-to-one functions, then $(g \circ f) : X \to Z$ is one-to-one.
(2) If $f : X \to Y$ and $g : Y \to Z$ are onto functions, then $(g \circ f) : X \to Z$ is onto.
(3) If R is a relation on X, then R is symmetric if and only if $R = R^{-1}$.
(4) If R is a reflexive relation on X, then R is transitive if and only if $R \circ R = R$.
(5) If X is a subset of Y then X and Y are not disjoint.

(1) T
(2) T
(3) T
(4) T
(5) T

$x R x$.

Date: October 13, 2018; Ramsey.

Exercise 0.2. Show that for all natural numbers n,

$$\sum_{i=1}^{n}(i+1)2^i = n2^{n+1}.$$

base case:

$n=1.$

$$(1+1)2^1 = 1\cdot 2^{(1+1)}$$

$$2\cdot 2 = 2^2$$

$$4=4$$

induction step:

suppose it's true for $n=k$.

$$LHS = k2^{k+1} + (k+2)2^{k+1}$$

$$= (2k+2)\cdot 2^{k+1}$$

$$= 2(k+1)\cdot 2^{k+1}$$

$$= (k+1)\cdot 2^{k+2}$$

$$= (k+1)\cdot 2^{(k+1+1)} = RHS \Rightarrow \text{it's true for } n=k+1.$$

So it's true for all natural numbers n.

Exercise 0.3. If B_1, B_2, C_1, C_2 are sets and $B_1 \subseteq C_1$ and $B_2 \subseteq C_2$ then
$$B_1 \cup B_2 \subseteq C_1 \cup C_2.$$

$B_1 \subseteq C_1$

$\Rightarrow B_1 \cup B_2 \subseteq C_1 \cup B_2.$

$B_2 \subseteq C_2$

$\Rightarrow C_1 \cup B_2 \subseteq C_1 \cup C_2$

so we have $B_1 \cup B_2 \subseteq C_1 \cup C_2.$

Exercise 0.4. Show that $n^2 - 7n + 13$ is nonnegative for all natural numbers $n \geq 3$.

base :

$n = 3$.

$3^2 - 7 \cdot 3 + 13 = 9 - 21 + 13 = 1 \geqslant 0$.

induction:

Suppose $k^2 - 7k + 13 \geq 0$.

$(k+1)^2 - 7(k+1) + 13$

$= k^2 + 2k + 1 - 7k - 7 + 13$

$= k^2 - 5k + 7$

$k^2 - 5k + 7 - (k^2 - 7k + 13)$

$= 2k - 6$

Because we have $k \geq 3$,

So $2k - 6 \geqslant 0$.

so $k^2 - 5k + 7 - (k^2 - 7k + 13) \geq 0$.

we also have $k^2 - 7k + 13 \geq 0$

So $k^2 - 5k + 7 \geq 0$

so it's true for $k+1$.

Then $n^2 - 7n + 13$ is nonnegative for all natural numbers $n \geq 3$.

Exercise 0.6. Suppose X and Y are sets.

(1) Suppose E_1 and E_2 are equivalence relations on X and Y respectively. Define a relation E on $X \times Y$ by

$$(x_1, y_1)E(x_2, y_2) \iff x_1 E_1 x_2 \text{ and } y_1 E_2 y_2.$$

Show E is an equivalence relation.

(2) Let S be the set of equivalence classes of E. Show that

$$S = \{[x]_{E_1} \times [y]_{E_2} : x \in X, y \in Y\},$$

where $[x]_{E_1}$ is the equivalence class of x with respect to the equivalence relation E_1 and $[y]_{E_2}$ is the equivalence class of y with respect to the equivalence relation E_2.

(1)

First we need to prove that it is reflexive:

Because E_1 and E_2 are reflexive, we have $x E_1 x$ and $y E_2 y$ $(x \in X, y \in Y)$. so we have $(x, y) E (x, y)$., so E is reflexive.

Symmetric: Suppose we have $(x_1, y_1) E (x_2, y_2)$. we have. $x_1 E_1 x_2$, $x_2 E_1 x_1$,

Because E_1 and E_2 are symmetric, we have $y_1 E_2 y_2$, $y_2 E_2 y_1$ $(x_1, x_2 \in X.\ y_1, y_2 \in Y)$. so we have $\begin{cases} x_2 E_1 x_1 \\ y_2 E_2 y_1 \end{cases} \Rightarrow (x_2, y_2) E (x_1, y_1)$

so E is symmetric.

Transitive: Suppose we have $(x_1, y_1) E (x_2, y_2)$ and $(x_2, y_2) E (x_3, y_3)$

Then we have $\left.\begin{cases} x_1 E_1 x_2,\ x_2 E_1 x_3 \Rightarrow x_1 E_1 x_3 \\ y_1 E_2 y_2,\ y_2 E_2 y_3 \Rightarrow y_1 E_2 y_3 \end{cases}\right\} \Rightarrow (x_1, y_1) E (x_3, y_3)$.

so E is transitive. So E is an equivalence relation.

(2).

An equivalence class of E is of the form $[(x_1, y_1)]_E = \{(x_2, y_2) \mid (x_1, y_1) E (x_2, y_2)$

so (x_2, y_2) has to fullfill $x_1 E_1 x_2$ and $y_1 E_2 y_2$, which means $x_2 \in [x_1]_{E_1}$ and $y_2 \in [y_1]_{E_2}$. So (x_2, y_2) is $[x_1]_{E_1} \times [y_1]_{E_2}$,

Because (x_1, y_1) is arbitrary, so it's true for all (x, y).

Then the set of equivalence classes of E should be $\{[x]_{E_1} \times [y]_{E_2} : x \in X, y \in Y\}$.

Exercise 0.7. Define a sequence by $t_1 = 2$ and $t_n = \prod_{i=1}^{n-1} t_i$ for all $n \geq 2$. Define an additional sequence by $s_n = \sum_{i=1}^{n} t_i$. Calculate s_3 and t_4.

$t_2 = \prod_{i=1}^{1} t_i = t_1 = 2.$

$t_3 = \prod_{i=1}^{2} t_i = t_1 \cdot t_2 = 4.$

$t_4 = \prod_{i=1}^{3} t_i = t_1 \cdot t_2 \cdot t_3 = 2 \cdot 2 \cdot 4 = 16.$

$$S_3 = \sum_{i=1}^{3} t_i = t_1 + t_2 + t_3$$
$$= 2 + 2 + 4$$
$$= 8.$$