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MATH 61 - MIDTERM EXAM 1

0.1. **Instructions.** This is a 50 minute exam. You should feel free to quote any theorems proved in class, as well as anything proved in the homework or discussion section. There are 7 questions—on the real exam, you are required to do the first true/false question, and choose 5 of the remaining 6. Only 5 problems other than the true/false question will be graded so *you should indicate which problems you want graded*, in the case that you attempt all 6. Each question is worth 10 points. Unless otherwise specified, you are required to justify your answers.

Grade circled ones

Exercise 0.1. Indicate whether the following statements are true or false. You do ~~not need to~~ justify your answer.

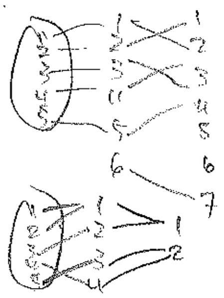
(1) If $f : X \to Y$ and $g : Y \to Z$ are one-to-one functions, then $(g \circ f) : X \to Z$ is one-to-one.

(2) If $f : X \to Y$ and $g : Y \to Z$ are onto functions, then $(g \circ f) : X \to Z$ is onto.

(3) If R is a relation on X, then R is symmetric if and only if $R = R^{-1}$.

(4) If R is a reflexive relation on X, then R is transitive if and only if $R \circ R = R$.

(5) If X is a subset of Y then X and Y are not disjoint.



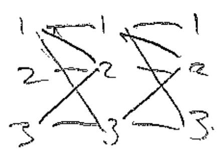
(1) T

(2) T

(3) T

(4) F

(5) F if $x = \{\emptyset\}$



(1,1) (2,1) (3,1)

~~(1,2)~~ ~~(2,1)~~

(1,2)

(x,y) $x \neq y$

Date: October 13, 2018; Ramsey.

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Exercise 0.2. Show that for all natural numbers n,

$$\sum_{i=1}^{n}(i+1)2^i = n2^{n+1}.$$

we can use induction.

Base case: $n=1$

$$\sum_{i=1}^{1}(i+1)2^i = 1 \cdot 2^{1+1}$$

$$\downarrow$$

$$2 \cdot 2^1 = 1 \cdot 2^2$$

$$4 = 4 \checkmark$$

Assume for some natural k

$$\sum_{i=1}^{k}(i+1)2^i = k \cdot 2^{k+1}$$

WTS:

$$\sum_{i=1}^{k+1}(i+1)2^i = (k+1)\,2^{(k+1)+1} = (k+1)\,2^{k+2}$$

$$\downarrow$$

$$= \sum_{i=1}^{k}(i+1)2^i + (k+2)2^{k+1}$$

$$\downarrow$$

$$= k(2^{k+1}) + (k+2)2^{k+1}$$

$$= (2^{k+1})(k+k+2)$$

$$= (2^{k+1})(2k+2)$$

$$= (k+1)(2^{k+2})$$

Thus, by mathematical induction, we are done.

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Exercise 0.3. If B_1, B_2, C_1, C_2 are sets and $B_1 \subseteq C_1$ and $B_2 \subseteq C_2$ then Prove $B_1 \cup B_2 \subseteq C_1 \cup C_2$.

Let $x \in B_1 \cup B_2$. Then,

$x \in B_1$ or $x \in B_2$ or both. (def of intersection).

If $x \in B_1$, $x \in C_1$ since $B_1 \subseteq C_1$. (definition of subset)

If $x \in B_2$, $x \in C_2$ since $B_2 \subseteq C_2$

since $C_1 \subseteq C_1 \cup C_2$ and $C_2 \subseteq C_1 \cup C_2$,

$x \in C_1 \cup C_2$. Thus, for any x in $B_1 \cup B_2$,

it must be in $C_1 \cup C_2$ so $B_1 \cup B_2 \subseteq C_1 \cup C_2$.

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Exercise 0.4. Show that $n^2-7n+13$ is nonnegative for all natural numbers $n \geq 3$.

We can use induction. nonnegative is ≥ 0

Base case: $n = 3$

$$3^2-7\cdot 3+13 = 9-21+13 = 1 \geq 0 \checkmark$$

Assume ~~there~~ is a natural number k s.t. $k \geq 3$

$$k^2-7k+13 \geq 0.$$

WTS:

$$(k+1)^2-7(k+1)+13 \geq 0$$

$$k^2+2k+1-7k-7+13 \geq 0$$

$$k^2-7k+13 + 2k+1-7 \geq 0$$

Since $k^2-7k+13 \geq 0$, it suffices to show $2k+1-7 \geq 0 \Rightarrow 2k-6 \geq 0$.

Since $k \geq 3$ (from problem statement),

$2k \geq 6 \Rightarrow 2k-6 \geq 0$, so we are done.

By induction, $n^2-7n+13$ is nonnegative for all natural numbers $n \geq 3$.

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Exercise 0.5. Let $f : X \to Y$ be a function. Given any subset $S \subseteq X$, we write $f(S)$ for the set defined as follows: $T \subseteq X$

$$f(S) = \{y \in Y : \text{ there is } s \in S \text{ such that } f(s) = y\}.$$

(1) Show that $f(S \cap T) \subseteq f(S) \cap f(T)$.
(2) Show that if f is injective, then $f(S \cap T) = f(S) \cap f(T)$.

(1) Let $y \in f(S \cap T)$ and by definition, $y \in Y$

there is some $x \in S \cap T$ s.t. $f(x) = y$.

If $x \in S \cap T$, $x \in S$ and $x \in T$ by definition of intersection

Then if $x \in S$ and $f(x) = y \Rightarrow y \in f(S)$ by definition of f

If $x \in T$ and $f(x) = y \Rightarrow y \in f(T)$ by definition of f.

since $y \in f(S)$ and $y \in f(T)$
$y \in f(S) \cap f(T)$ by definition of intersection

Thus, for any $y \in f(S \cap T)$, $y \in f(S) \cap f(T)$

so $f(S \cap T) \subseteq f(S) \cap f(T)$ ☐

(2) To show $f(S \cap T) = f(S) \cap f(T)$, we must prove $f(S \cap T) \subseteq f(S) \cap f(T)$ (done by (1))
and $f(S \cap T) \supseteq f(S) \cap f(T)$.
Let $y \in f(S) \cap f(T) \Rightarrow y \in f(S)$ and $y \in f(T)$
then, there exists $f(x) = y$ where $x \in S$ and $x \in T$.
Since $x \in S$ and $x \in T$, $x \in S \cap T$. But this implies
$y \in f(S \cap T)$ since there is an x s.t. $x \in S \cap T$
and $f(x) = y$. Thus, any element in $f(S) \cap f(T)$
is a part of $f(S \cap T)$ and vice versa
$\Rightarrow f(S \cap T) = f(S) \cap f(T)$.

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~~Exercise~~ 0.6. Suppose X and Y are sets.

(1) Suppose E_1 and E_2 are equivalence relations on X and Y respectively. Define a relation E on $X \times Y$ by

$$(x_1, y_1)E(x_2, y_2) \iff x_1E_1x_2 \text{ and } y_1E_2y_2.$$

Show E is an equivalence relation.

(2) Let $\mathcal{S}$ be the set of equivalence classes of E. Show that

$$\mathcal{S} = \{[x]_{E_1} \times [y]_{E_2} : x \in X, y \in Y\},$$

where $[x]_{E_1}$ is the equivalence class of x with respect to the equivalence relation E_1 and $[y]_{E_2}$ is the equivalence class of y with respect to the equivalence relation E_2.

(1) Reflexivity: WTS $(x,y)E(x,y)$

WTS => xE_1x and yE_2y

since E_1, E_2 are equiv. relations,

both have reflexivity so xE_1x and yE_2y

=> $(x,y)E(x,y)$.

Symmetric: Let there exist x_1, y_1, x_2, y_2

such that $(x_1,y_1)E(x_2,y_2)$.

Then $x_1E_1x_2$ and $y_1E_2y_2$ by definition.

Since E_1, E_2 are symmetric, $x_2E_1x_1$ and $y_2E_2y_1$,

which means $(x_2,y_2)E(x_1,y_1)$.

Transitive: Let there be $x_1, y_1, x_2, y_2, x_3, y_3$ s.t.

$(x_1,y_1)E(x_2,y_2)$ and $(x_2,y_2)E(x_3,y_3)$.

This implies: $x_1E_1x_2$ and $x_2E_1x_3$ => $x_1E_1x_3$

and $y_1E_2y_2$ and $y_2E_2y_3$ => $y_1E_2y_3$

since E_1, E_2 are transitive.

Since $x_1E_1x_3$ and $y_1E_2y_3$, $(x_1,y_1)E(x_3,y_3)$

so E is transitive.

E is reflexive, symmetric, and transitive,

so it is an equivalence relation.

(2) On another page.

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Exercise 0.7. Define a sequence by $t_1 = 2$ and $t_n = \prod_{i=1}^{n-1} t_i$ for all $i \geq 2$. Define an additional sequence by $s_n = \sum_{i=1}^{n} t_i$. Calculate s_3 and t_4.

$t_1 = 2$

$t_2 = t_1 = 2$

$t_3 = t_2 \cdot t_1 = 4$

$t_4 = t_3 \cdot t_2 \cdot t_1 = 16$

$s_1 = t_1 = 2$

$s_2 = t_1 + t_2 = 4$

$s_3 = t_1 + t_2 + t_3 = 8$

$\boxed{s_3 = 8 \, , \, t_4 = 16}$