61 Midterm 1

TOTAL POINTS

54/60

QUESTION 1

1 Exercise 1 **6 / 10**

- **- 2 pts** (1) Incorrect
- **- 2 pts** (2) Incorrect
- ✓ **- 2 pts (3) Incorrect**
- **- 2 pts** (4) Incorrect
- ✓ **- 2 pts (5) Incorrect**
- **- 0 pts** Fully correct

5) Let X be the empty set.

QUESTION 2

2 Exercise 2 **10 / 10**

- **- 8 pts** Only basis case correct
- **- 8 pts** only demonstrated understanding of Sigma notation
- **- 4 pts** Did not complete inductive step correctly
- **- 2 pts** algebra/logic errors
- **- 1 pts** Minor arithmetic errors
- **- 1 pts** Structure of the proof not made clear (e.g. induction hypothesis not mentioned explicitly, or not clear how induction hypothesis was used).
- ✓ **- 0 pts fully correct**
- **- 10 pts** Not graded

QUESTION 3

3 Exercise 3 **10 / 10**

- **- 10 pts** Incorrect without partial solution
- **- 8 pts** Drew Venn diagram correct but written proof incorrect, or some informal argument
- **- 8 pts** Gives an argument that only works for finite sets
- **- 8 pts** Mixes up union with intersection or Cartesian product
- **- 5 pts** Contains parts of a valid formal approach
- ✓ **- 0 pts Fully correct**
- **- 10 pts** Skipped

It is not needed to show that the two unions are not always equal.

QUESTION 4

4 Exercise 4 **10 / 10**

- ✓ **- 0 pts Completely correct**
- **- 3 pts** Incorrect basis case
- **- 4 pts** Correctly implemented induction but didn't succeed in inductive step
- **- 1 pts** Minor arithmetic errors
- **- 6 pts** Correct basis case but did not set up inductive step
- **- 10 pts** skipped

QUESTION 5

5 Exercise 5 **0 / 10**

- ✓ **- 10 pts Not graded**
- **- 5 pts** (1) Incorrect/blank
- **- 4 pts** (1) minimal progress
- **- 4 pts** (1) Diagram or example only
- **- 3 pts** (1) Vague explanation/missing or incorrect steps
- **- 1 pts** (1) Mostly correct, but lacking detail, or disorganized.
- **- 1 pts** (1) Used "X subset Y --> f(X) subset f(Y)" without proof
- **- 5 pts** (2) Incorrect/blank
- **- 4 pts** (2) minimal progress (e.g. just writing the definition of injective.)
- **- 3 pts** (2) Vague explanation/missing steps
- **- 2 pts** (2) Partially correct, but some steps missing
- **- 1 pts** (2) Mostly correct, but unclear or lacking detail.
- **- 0 pts** fully correct

QUESTION 6

6 Exercise 6 **8 / 10**

- **- 0 pts** correct
- **- 2 pts** (1) Demonstrated understanding of problem but did not give correct solution
- **- 1 pts** (1) Failed to show all 3 conditions of an equivalence rel
- ✓ **- 2 pts (2) Incorrect solution but demonstrated understanding of equivalence class**
- **- 5 pts** (1) Blank
- **- 5 pts** (2) Blank
- **- 10 pts** Not Graded
- **- 8 pts** Incorrect and did not demonstrate understanding of problem

QUESTION 7

7 Exercise 7 **10 / 10**

- ✓ **- 0 pts Correct**
- **- 2 pts** Arithmetic errors
- **- 10 pts** Incorrect answer with no work
- **- 10 pts** Not graded
- **- 5 pts** Miscalculate tn

Grade: 2, 3, 4, 6, 7

MATH 61 - MIDTERM EXAM 1

0.1. **Instructions.** This is a 50 minute exam. You should feel free to quote any theorems proved in class, as well as anything proved in the homework or discussion section. There are 7 questions—on the real exam, you are required to do the first true/false question, and choose 5 of the remaining 6. Only 5 problems other than the true/false question will be graded so *you should indicate which problems you want graded*, in the case that you attempt all 6. Each question is worth 10 points. Unless otherwise specified, you are required to justify your answers.

Exercise 0.1. Indicate whether the following statements are true or false. You do not need to justify your answer.

(1) If $f : X \to Y$ and $g : Y \to Z$ are one-to-one functions, then $(g \circ f) : X \to Z$ is one-to-one.
(2) If $f : X \to Y$ and $g : Y \to Z$ are onto functions, then $(g \circ f) : X \to Z$ is onto.
(3) If R is a relation on X, then R is symmetric if and only if $R = R^{-1}$.
(4) If R is a reflexive relation on X, then R is transitive if and only if $R \circ R = R$.
(5) If X is a subset of Y then X and Y are not disjoint.

(1) True
(2) True
(3) False
(4) True
(5) True

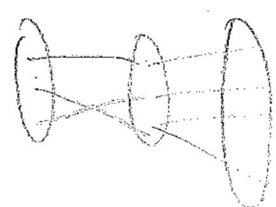
(1, 1) (2, 2) (3, 3), (1, 2), (2, 3), (1, 3)

(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)

$\ln(x)$

x^3

$\ln(x^3)$



Date: October 13, 2018; Ramsey.

Exercise 0.2. Show that for all natural numbers n,

$$\sum_{i=1}^{n}(i+1)2^i = n2^{n+1}.$$

Basis: $n=1$

$$(1+1)2^1 = 4 = (1)2^2$$

Induction: Assume true for K $\left(\sum_{i=1}^{k}(i+1)2^i = k2^{k+1}\right)$

Show true for $k+1$

LHS: $$\sum_{i=1}^{k+1}(i+1)2^i = \sum_{i=1}^{k}(i+1)2^i \cdot (k+2)2^{k+1}$$

RHS: $(k+1)\,2^{k+2}$

$$= (k)2^{k+1} + (k+2)2^{k+1}$$

$$= 2^{k+1}(2k+2)$$

$$= 2 \cdot 2^{k+1}(k+1)$$

$$= (k+1)2^{k+2}$$

equal

Therefore by mathematical induction, $\sum_{i=1}^{n}(i+1)2^i = n2^{n+1}$ for all natural numbers n.

Exercise 0.3. If B_1, B_2, C_1, C_2 are sets and $B_1 \subseteq C_1$ and $B_2 \subseteq C_2$ then
Prove
$$B_1 \cup B_2 \subseteq C_1 \cup C_2.$$

If $x \in B_1$, then $x \in C_1$ and thus
$x \in B_1 \cup B_2$ and $x \in C_1 \cup C_2$

Similarly if $x \in B_2$, then $x \in C_2$ and thus
$x \in B_1 \cup B_2$ and $x \in C_1 \cup C_2$

$\therefore$ we know $B_1 \cup B_2 \subseteq C_1 \cup C_2$ but we must show it doesn't hold vice versa

So if $x \in C_1$ or $x \in C_2$ we cannot say $x \in B_1$ or $x \in B_2$
Since C_1 and C_2 are not necessarily subsets of B_1 and B_2

Thus $B_1 \cup B_2 \subseteq C_1 \cup C_2$

Exercise 0.4. Show that $n^2 - 7n + 13$ is nonnegative for all natural numbers $n \geq 3$.
≥ 0

$n^2 - 7n + 13 \geq 0$ for $n \geq 3$

Basis: $n = 3$

$(3)^2 - 7(3) + 13 = 9 - 21 + 13 = 1$

$1 \geq 0$ ✓

Induction: Assume true for k $(k^2 - 7k + 13 \geq 0)$

Show true for $k+1$

$\underbrace{(k+1)^2 - 7(k+1) + 13}$ ≥ 0

$= k^2 + 2k + 1 - 7k - 7 + 13$

$= (k^2 - 7k + 13) + (2k - 6)$

We know $k^2 - 7k + 13$ is ≥ 0 because of our assumption, so we must show $2k - 6 \geq 0$

$2k - 6 \geq 0$

$2k \geq 6$

$k \geq 3$ ← this is true b/c we are given this in the problem that $n \geq 3$

$\therefore$ Both terms $(k^2 - 7k + 13)$ and $(2k - 6)$ are nonnegative, so by mathematical induction, we conclude that $n^2 - 7n + 13$ is nonnegative for all natural numbers $n \geq 3$.

Do not grade this question

Exercise 0.5. Let $f : X \to Y$ be a function. Given any subset $S \subseteq X$, we write $f(S)$ for the set defined as follows:

$$f(S) = \{y \in Y : \text{ there is } s \in S \text{ such that } f(s) = y\}.$$

(1) Show that $f(S \cap T) \subseteq f(S) \cap f(T)$.
(2) Show that if f is injective, then $f(S \cap T) = f(S) \cap f(T)$.

Exercise 0.6. Suppose X and Y are sets.

(1) Suppose E_1 and E_2 are equivalence relations on X and Y respectively. Define a relation E on $X \times Y$ by
$$(x_1, y_1)E(x_2, y_2) \iff x_1 E_1 x_2 \text{ and } y_1 E_2 y_2.$$
Show E is an equivalence relation.

(2) Let S be the set of equivalence classes of E. Show that
$$S = \{[x]_{E_1} \times [y]_{E_2} : x \in X, y \in Y\},$$
where $[x]_{E_1}$ is the equivalence class of x with respect to the equivalence relation E_1 and $[y]_{E_2}$ is the equivalence class of y with respect to the equivalence relation E_2.

(1.) Reflexive: Show $(x,y)E(x,y)$

$\Rightarrow$ xE_1x and yE_2y which is true since E_1 and E_2 are reflexive. Therefore $(x,y)E(x,y)$ So E is reflexive.

Symmetric: Show $(x_1,y_1)E(x_2,y_2)$ and $(x_2,y_2)E(x_1,y_1)$

$\Rightarrow$ $x_1E_1x_2$ and $x_2E_1x_1$ $\quad (x_1,y_1)E(x_2,y_2) \rightarrow$ symmetric

$y_1E_2y_2$ and $y_2E_2y_1 \rightarrow (x_2,y_2)E(x_1,y_1)$

Since E_1 and E_2 are symmetric, this holds and so E is symmetric as well

Transitive: Show if $(x_1,y_1)E(x_2,y_2)$ and $(x_2,y_2)E(x_3,y_3)$, then $(x_1,y_1)E(x_3,y_3)$

$\Rightarrow$ $x_1E_1x_2$, $y_1E_2y_2$ $\xRightarrow{\text{transitivity of } E_1 \text{ and } E_2}$ $x_1E_1x_3$ and $y_1E_2y_3$

$x_2E_1x_3$, $y_2E_2y_3$

$\therefore$ $(x_1,y_1)E(x_3,y_3)$ and so E is transitive

(2) Suppose $x_1 \in [x]_{E_1}$ and $y_2 \in [y]_{E_2}$

$x_2 \in [x]_{E_1}$ $\quad y_1 \in [y]_{E_2}$

$\Rightarrow$ $x_1E_1x_2$ and $y_1E_2y_2$

which means that $(x_1,y_1)E(x_2,y_2)$, so (x_1,y_1) and (x_2,y_2) must be in the same equivalence class for E. Thus, this equivalence class holds the points (x_1,y_1) and (x_2,y_2), but since $x_1,x_2 \in [x]_{E_1}$ and $y_1/y_2 \in [y]_{E_2}$, this can be written as $([x]_{E_1}, [y]_{E_2})$ which is the same as $[x]_{E_1} \times [y]_{E_2}$.

Exercise 0.7. Define a sequence by $t_1 = 2$ and $t_n = \prod_{i=1}^{n-1} t_i$ for all $i \geq 2$. Define an additional sequence by $s_n = \sum_{i=1}^{n} t_i$. Calculate s_3 and t_4.

$t_1 = 2$. $t_n = \prod_{i=1}^{n-1} t_i$. for all $n \geq 2$

$S_n = \sum_{i=1}^{n} t_i$

$t_1 = 2$

$t_2 = \prod_{i=1}^{1} t_i = 2$

$t_3 = \prod_{i=1}^{2} t_i = 2 \cdot 2 = 4$

$t_4 = \prod_{i=1}^{3} t_i = 2 \cdot 2 \cdot 4 = 16$

$\boxed{t_4 = 16}$

$S_3 = \sum_{i=1}^{3} t_i = t_1 + t_2 + t_3 = 2 + 2 + 4 = 8$

$\boxed{S_3 = 8}$