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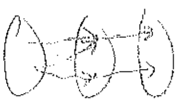
MATH 61 - MIDTERM EXAM 1

0.1. **Instructions.** This is a 50 minute exam. You should feel free to quote any theorems proved in class, as well as anything proved in the homework or discussion section. There are 7 questions—on the real exam, you are required to do the first true/false question, and choose 5 of the remaining 6. Only 5 problems other than the true/false question will be graded so *you should indicate which problems you want graded*, in the case that you attempt all 6. Each question is worth 10 points. Unless otherwise specified, you are required to justify your answers.

Exercise 0.1. Indicate whether the following statements are true or false. You do not need to justify your answer.

(1) If $f : X \to Y$ and $g : Y \to Z$ are one-to-one functions, then $(g \circ f) : X \to Z$ is one-to-one.
(2) If $f : X \to Y$ and $g : Y \to Z$ are onto functions, then $(g \circ f) : X \to Z$ is onto.
(3) If R is a relation on X, then R is symmetric if and only if $R = R^{-1}$.
(4) If R is a reflexive relation on X, then R is transitive if and only if $R \circ R = R$.
(5) If X is a subset of Y then X and Y are not disjoint.



1) True

2) True

$x R y \Leftrightarrow y R^{-1} x$ 3) True

$y R x \Leftrightarrow x R^{-1} y$ 4) False

5) True

$\{(1,1),(2,2)\}$

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Exercise 0.2. Show that for all natural numbers n,

$$\sum_{i=1}^{n}(i+1)2^i = n2^{n+1}.$$

$n=1$: $\sum_{i=1}^{1}(i+1)2^i = 2\,2^1 = 4 = 1\cdot 2^2$

$n=k$: $\sum_{i=1}^{k}(i+1)2^i = k2^{k+1}$ by def

$n=k+1$: $\sum_{i=1}^{k+1}(i+1)2^i$

$= \sum_{i=1}^{k}(i+1)2^i + (k+2)2^{k+1}$

$= k2^{k+1} + (k+2)2^{k+1}$

$k2^{k+1} + k2^{k+1} + 2^{k+2}$

$= 2k2^{k+1} + 2^{k+2}$

$= k2^{k+2} + 2^{k+2}$

$= (k+1)2^{k+2}$

Thus by induction, the equality is true $\forall n \in \mathbb{N}$

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Exercise 0.3. If B_1, B_2, C_1, C_2 are sets and $B_1 \subsetneq C_1$ and $B_2 \subseteq C_2$ then prove $B_1 \cup B_2 \subseteq C_1 \cup C_2$.

$B_1 \subseteq C_1 \quad B_2 \subseteq C_2$

$B_1 \cup B_2 \subseteq C_1 \cup C_2$

Take a $b \in B_1 \cup B_2$

Case 1: $b \in B_1$, then $b \in C_1$ b/c $B_1 \subseteq C_1$

Case 2: $b \in B_2$, then $b \in C_2$ b/c $B_2 \subseteq C_2$

Then $\forall\, b \in B_1 \cup B_2$, it lies within C_1 or C_2 as well

so $B_1 \cup B_2 \subseteq C_1 \cup C_2$

(since $C_1 \cup C_2$ by def of union contains all elements of either C_1 or C_2)

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Exercise 0.4. Show that $n^2-7n+13$ is nonnegative for all natural numbers $n \geq 3$.

$n=3$: $9-21+13=1$ ✓

$n=k$: $k^2-7k+13$ is nonnegative by def

$n=k+1$: $(k+1)^2-7(k+1)+13$

$= k^2+2k+1-7k-7+13$

$= \underbrace{(k^2-7k+13)}_{\text{non neg}} + \underbrace{(2k-6)}_{\text{non neg}}$ when $k \geq 3$

since $2(3)-6=0$

then $2(k+1)-6=(2k-6)+2 > 0$ for $k \geq 3$

(by induction)

two non neg terms added together = non neg

and so by induction, the claim holds $\forall\, n \geq 3$

Exercise 0.5. Let $f : X \to Y$ be a function. Given any subset $S \subseteq X$, we write $f(S)$ for the set defined as follows: $T \subseteq X$

$$f(S) = \{y \in Y : \text{ there is } s \in S \text{ such that } f(s) = y\}.$$ $y = f(s)$

(1) Show that $f(S \cap T) \subseteq f(S) \cap f(T)$.
(2) Show that if f is injective, then $f(S \cap T) = f(S) \cap f(T)$.

1) Choose a $s \in S \cap T$

then $f(s) \in f(S \cap T)$;

also, since $s \in S$ and $s \in T$

then $f(s) \in f(S)$ and $f(s) \in f(T)$

so $f(s) \in f(S) \cap f(T)$

then

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Exercise 0.6. Suppose X and Y are sets.

(1) Suppose E_1 and E_2 are equivalence relations on X and Y respectively. Define a relation E on $X \times Y$ by
$$(x_1, y_1)E(x_2, y_2) \iff x_1E_1x_2 \text{ and } y_1E_2y_2.$$
Show E is an equivalence relation.

(2) Let S be the set of equivalence classes of E. Show that
$$S = \{[x]_{E_1} \times [y]_{E_2} : x \in X, y \in Y\},$$
where $[x]_{E_1}$ is the equivalence class of x with respect to the equivalence relation E_1 and $[y]_{E_2}$ is the equivalence class of y with respect to the equivalence relation E_2.

1) Reflexive: $(x_1, y_1) E (x_1, y_1) \Rightarrow x_1 E_1 x_1$ and $y_1 E_2 y_1$
which is true since E_1, E_2 are equiv. relations

Transitive: $(x_1, y_1) E (x_2, y_2) \Rightarrow x_1 E_1 x_2$ and $y_1 E_2 y_2$
$(x_2, y_2) E (x_3, y_3) \Rightarrow x_2 E_1 x_3$ and $y_2 E_2 y_3$
then $x_1 E_1 x_3$ and $y_1 E_2 y_3$
$(x_1, y_1) E (x_3, y_3)$ ✓

Symmetric: $(x_1, y_1) E (x_2, y_2) \Rightarrow x_1 E_1 x_2$ and $y_1 E_2 y_2$
implies $\Downarrow$ implies $\Downarrow$
$x_2 E_1 x_1$ and $y_2 E_2 y_1$
$(x_2, y_2) E (x_1, y_1)$ ✓

2) If S = set of all equiv. classes of E, then
$S = \{[(x,y)]_E \mid x \in X, y \in Y\}$
if $(x_1, y_1) \in [(x,y)]_E$
then $[(x,y)]_E = [(x_1, y_1)]_E$
since x and y are splittable into components, (since x and y are independent, their relations are independent)
then $[x]_{E_1} = [x_1]_{E_1}$ and $[y]_{E_2} = [y_1]_{E_2}$

and so $S = \{[x]_{E_1} \times [y]_{E_2} \mid x \in X, y \in Y\}$
$= \{[(x,y)]_E \mid x \in X, y \in Y\}$

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Exercise 0.7. Define a sequence by $t_1 = 2$ and $t_n = \prod_{i=1}^{n-1} t_i$ for all $i \geq 2$. Define an additional sequence by $s_n = \sum_{i=1}^{n} t_i$. Calculate s_3 and t_4.

$t_1 = 2$

$t_2 = \prod_{i=1}^{1} t_i = t_1 = 2$

$t_3 = \prod_{i=1}^{2} t_i = t_1 \cdot t_2 = 2 \cdot 2 = 2^2$

$t_4 = \prod_{i=1}^{3} t_i = t_1 \cdot t_2 \cdot t_3 = 2 \cdot 2 \cdot 2^2 = 16$

$s_3 = \sum_{i=1}^{3} t_i = t_1 + t_2 + t_3 = 2 + 2 + 4 = 8$