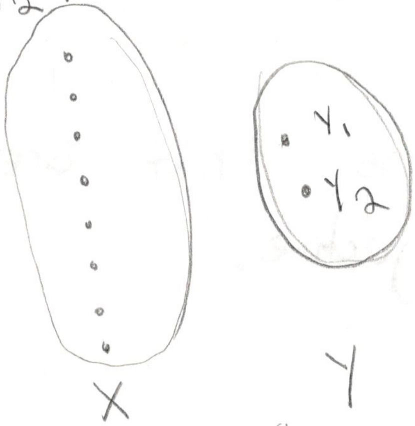
(1)

(1) each $x \in X$ has 2 choices for a y; y_1 and y_2. There are 8 $x \in X$

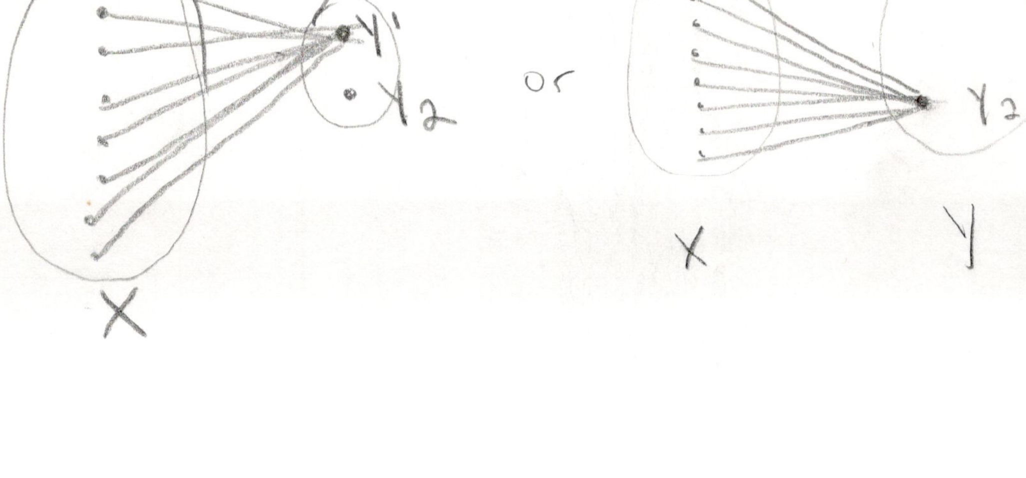


$2^8 = 256$

(2)

1. How many are not injective?
All. There is no case in which this function is injective. In order to be injective each element in Y would have at most 1 arrow pointing to it. We need to map 8 $x \in X$ onto 2 $y \in Y$. Using Pigeonhole Principle, if $n=8$ (pigeons) and $k=2$ (pigeonholes), where $k<n$, some k contain at least 2 n (meaning some $y \in Y$ will have at least 2 arrows)

1. How many are not surjective?
There are 2 cases in which $y \in Y$ has <1 arrows; either y_1 has zero and y_2 has 8 or y_2 has 0 and y_1 has 8.



(1)

All functions

not Injective

not injective
not surjective

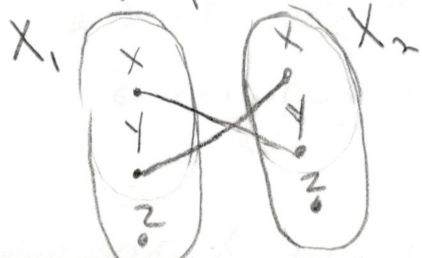
So there are 2 functions that are neither injective nor surjective.

② Say $x \in X$ and $f(x)=y$ where $y \in X$

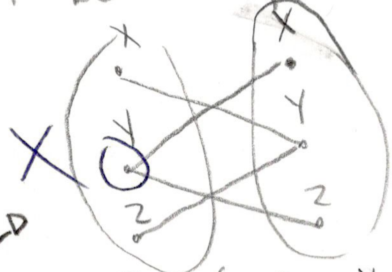
There are 2 cases in which $f(f(x))=x$ are satisfied by a function. (Say set 1 = X_1, set 2 = X_2)

Case 1:

$f(x)=y$ where $x \neq y$. That must mean $f(y)=x$.



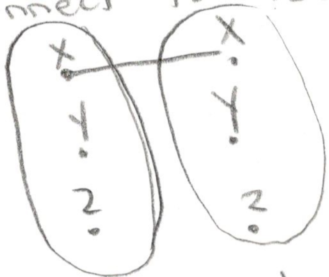
A third element, say $z \in X$, cannot be assigned to the members x or $y \in X_2$. This is because if $f(z)=y$ then $\underline{f(y)=z}$, and since $f: X \to X$ is a function, y cannot be assigned to more than one member.



so if $f(x)=y$, then $f(z) \neq y$ where $x \neq z$

Case 2:

$f(x)=y$ where $x=y$. This is equivalent to $f(x)=x$. if $f(x)=x$ then $f(f(x))=f(x)=x$. ✓ This shows $x \in X_1$ can connect to itself in $x \in X_2$.



Additionally no outside element can be assigned to x now, because $f(y)=x$ and $\underline{f(x)=y}$ would have to be true, and again this would make it not a function.

these are the only 2 cases of assignment in this function, and they both prohibit the assignment of 2 different $x \in X_1$ to the same $x \in X_2$. Additionally, because the function is $X \to X$, there is the same number of elements in X_1 as there are in X_2. These two facts together mean there is a one to one correspondence, meaning this function is bijective.

(Alternitively there are X pigeons, X pigeonholes and you cannot have ≥ 1 pigeons in the same hole).

3

1. When $|X|=1$, there can only be one equivalence class on one equipartition

$[1] = \{1\}$ | This being because You can only connect x to itself

When $|X| = 2$, 2 equipartitions can be made

one that has 1 equivalence class [all vertices are connected] and one that has $|X|$ equivalence classes, each the size of one.

If X increases by 1, this remains true, because adding a vertex does not change the fact that the equivalence relation could have one equipartition where the number of classes is one and class size $= |X+1|$ and another where each vertex is an equivalence class, so $|X+1| =$ # of classes and class size $= 1$.

2.

9 can be divided into equally sized classes in these ways:

1) 1 class of size 9

2) 3 classes of size 3

3) 9 classes of size 1

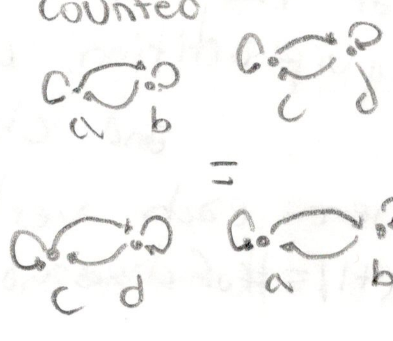
1 class of size 9 can have elements selected 1 way. equally 9 classes of size 1 can also have elements selected 1 way

3 classes of size 3:

$$\frac{9!}{(9-3)! \cdot 3! \cdot 3!} = 14$$

we divide by an extra 3! for the number of classes which are over counted

(ex:



$\underline{14 + 1 + 1 = 16}$

(4) ALIVE
1 2 3 4 5

total arrangements − Arrangements with EVIL

5!

(2) cases

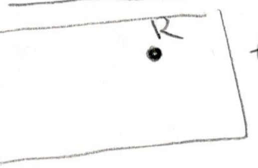
A+EVIL

EVIL+A

$5! - 2$

⑤ **Theorem:** a full splitting binary tree of height n has t terminal vertices, where $t = 2^n$

base case $n=0$

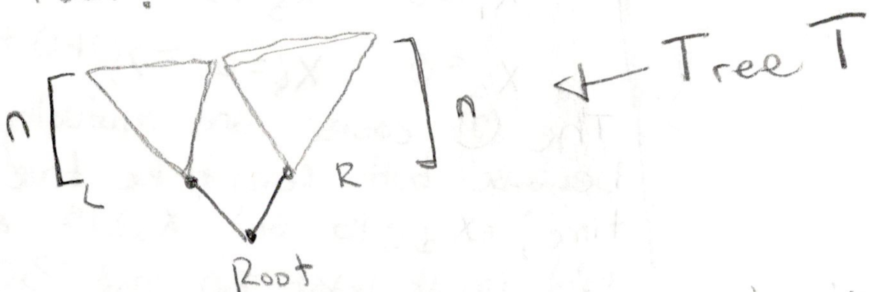


$t = 2^n$, $t = 2^0$ so $t=1$. This is true because with height zero, there is only 1 vertex (that being the root) and if there is only a root, the root is a terminal vertex

Inductive Step

Assuming $t = 2^n$, where n = height t = # terminal vertices for a full splitting binary tree, Consider the full splitting tree with height $n+1$.

Because this tree is full binary, there are two children of the root.



Since this tree is splitting, all terminal vertices are at the same height. In a binary tree, where height $= n+1$, at least one subtree rooted at v (where v is a child of the root) must have height n. If at least one child subtree must have height n, and all terminal vertices are same height, and this is a full binary so the root must have 2 children, then both child subtrees must have height n.

Since every terminal vertex of T is contained in a tree T rooted at a child of the root (and there are 2 of these trees because T is full binary)

$$\boxed{t = 2^n + 2^n = 2^{n+1}}$$

5 Since a Full splitting binary tree is a full binary, it has these properties:

$t = i + 1$ (i = internal vertices, t = terminal)

total vertices $= 2i + 1$

$t = 2^h$

$i = 2^h - 1$

total $= 2(2^h - 1) + 1$

$= 2^{h+1} - 2 + 1$

total $= 2^{h+1} - 1$ vertices

6) $x_1 + x_2 + x_3 + x_4 = 17$; x_1, x_2, x_3, x_4 non neg,

$x_1 \geq 2$

$x_2 \leq 14$

$x_3 \leq 14$

$x_4 \geq 0$

$x_1' = x_1 - 2$; $x_1' \geq 0$

$x_1' + x_2 + x_3 + x_4 = 15$ minus ($x_2 \geq 15$ or $x_3 \geq 15$)

$\binom{15+4-1}{15} = \binom{18}{15}$

① case where $x_2 \geq 15$

$x_1' = 0$ $x_3 = 0$ $\Rightarrow$ $0+15+0+0 = 15$

$x_2 = 15$ $x_4 = 0$

① case where $x_3 \geq 15$

$x_1' = 0$ $x_3 = 15$

$x_2 = 0$ $x_4 = 0$ $\Rightarrow 0+0+15+0 = 15$

The ② cases are mutually exclusive because both cannot be true at same time; $ex_2 \geq 15$ and $x_3 \geq 15$ means the total is at least 30 and $30 > 15$

$\boxed{\binom{18}{15} - 2}$

7)

1. To have an edge to (0,0), a vertex must meet 1 of 2 sets of requirments

   1. $|a-c|=1$ and $b=d$

   or

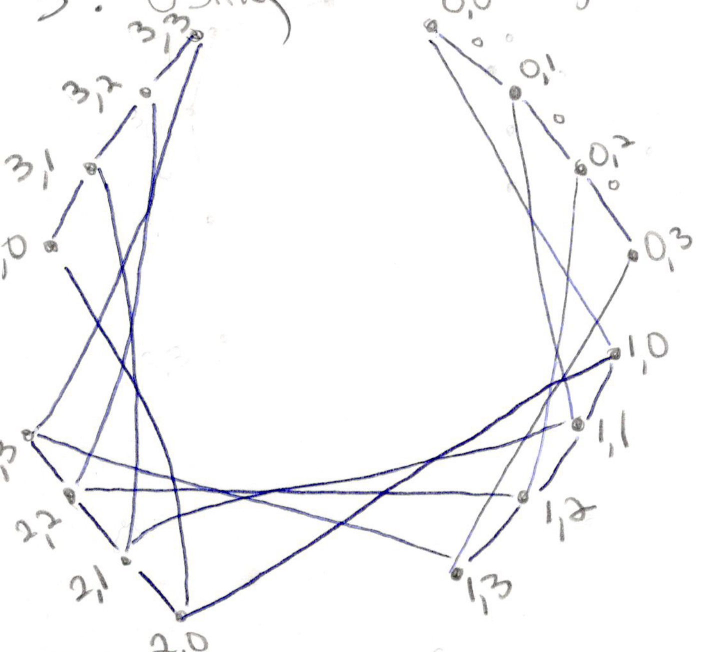
   2. $|b-d|=1$ and $a=c$

For all numbers, X is integer $0 \le X \le 3$

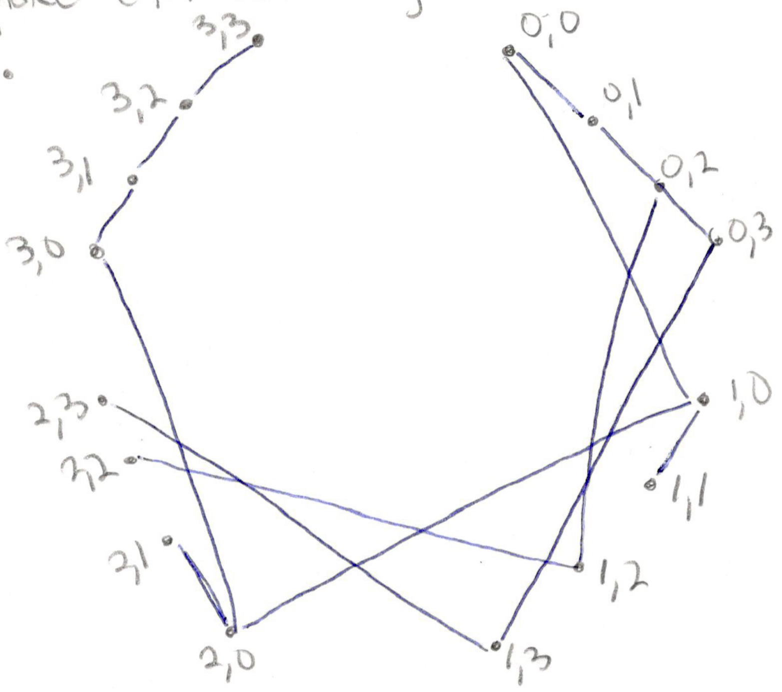
lets say $(0,0)=(a,b)$ and a vertex connected to (0,0) is (c,d). To meet set 1 requirments, $|0-c|=1$, so $c=-1,1$. c must be nonnegative, so c must equal 1. $b=d$, so d must equal 0. This means (1,0) is connected to (0,0). To meet set 2 requirments, $|0-d|=1$, so $d=1$ (since is only positive) and $a=c$ so $c=0$. This means (0,1) is also connected to (0,0). These 2 connections make (0,0) have degree 2.

2. Following the method expressed in part 1, say $(2,2)=(a,b)$ and a vertex connected to (a,b) is (c,d). to meet set 1 requirment s $|2-c|=1$ and $0 \le c \le 3$ so $c=1,3$. $b=d$ so $d=2$. This creates 2 unique (c,d) possibilities connected to (2,2); (1,2) and (3,2). To meet set 2 requirments, $|2-d|=1$ and $0 \le d \le 3$ so $d=1,3$. $a=c$ so $2=c$. This creates 2 more, unique, (c,d) possibilities connected to (2,2); (2,1) and (2,3). These 4 connections make (2,2) have degree 4.

3. Using Prim's algorithm:



Graph



Min Spanning tree

Adding edge values:

Weights: $e((0,0),(0,1)) = 0$

$e((0,1),(0,2)) = 0$

$e((0,2),(0,3)) = 0$

$e((0,3),(1,3)) = 0$

$e((0,2),(1,2)) = 0$

$e((0,0),(1,0)) = 0$

$e((1,0),(1,1)) = 0$

$e((1,0),(2,0)) = 0$

$e((2,0),(2,1)) = 0$

$e((2,0)(3,0)) = 0$

$e(3,0),(3,1)) = 0$

$e((1,3),(2,3)) = 1$

$e(1,2),(2,2)) = 1$

$+ \quad e(3,1),(3,2)) = 1$

$e(3,2),(3,3)) = 2$

---

weight $= 5$

8) Say we want to make a bipartite graph with 17 vertices and at least 9 vertices of degree 9. We make subsets V_1 and V_2. In bipartite, each vertex in V_1 can at most connect to each Vertex in V_2. Thus the maximum degree of a vertex in V_1 is $|V_2|$ and the max degree of a vertex in V_2 is $|V_1|$. We'd need a graph with these properties: $|V_1| \geq 9$ and $|V_2| \geq 9$ and $|V_1| + |V_2| = 17$. This is a contradiction, because the lowest values to fufill this arrangment $(|V_1| = 9,\ |V_2| = 9)$

$\underline{9 + 9 = 18 \neq 17}$.

So a graph G with 17 vertices and at least 9 vertices with degree 9 cannot be bipartite.