Math 61 Final Exam

JOE PINTO, SR

TOTAL POINTS

40 / 40

QUESTION 1

1 Counting functions **5 / 5**

✓ **+ 2 pts** **a: $2^8 = 256$ options total**

✓ **+ 3 pts** **b: 2 functions total, by casing on which element of Y is not in the image**

+ **0 pts** a: Incorrect count

+ **2 pts** b: Correct count, but did not properly justify some steps

+ **1.5 pts** b: Incorrect count, but some promising progress.

+ **0 pts** b: Incorrect count, as well as several other incorrect statements along the way

QUESTION 2

2 f squared equals the identity **5 / 5**

✓ **+ 2 pts** **Injectivity: Let $x, y \in X$ be such that $f(x) = f(y)$. Then $f(f(x)) = f(f(y))$. Thus $x = f(f(x)) = f(f(y)) = y$, so that $x = y$. Hence, for any $x, y \in X$, if $f(x) = f(y)$, we have $x = y$. We conclude f is injective, as desired.**

+ **1 pts** Injectivity: Partial progress, but informal or missing some important details

+ **0 pts** Injectivity: No serious progress. Arguments that may fall into this category include: trying to use f^{-1} without justifying f is invertible, trying to claim f is the identity function, trying to use the cardinality of X (e.g. assuming $|X|< \infty$), showing injectivity of $f \circ f$ instead of f

✓ **+ 3 pts** **Surjectivity: Let $y \in X$ be arbitrary. Notice $f(f(y)) = y$. Thus, there exists an $x \in X$, namely $x = f(y)$, such that $f(x) = y$. Since $y \in X$ was arbitrary, we see that for all $y \in X$, there exists an $x \in X$ such that $f(x) = y$. Thus, f is surjective, as desired.**

+ **1.5 pts** Surjectivity: Incorrect or missing quantifiers, but argument is correct or almost correct. An informal but (almost) correct argument may also fall into this category.

+ **0 pts** Surjectivity: No serious progress. Arguments that may fall into this category include: trying to use f^{-1} without justifying f is invertible, trying to claim f is the identity function, trying to use the cardinality of X (e.g. assuming $|X|< \infty$), showing surjectivity of $f \circ f$ instead of f

QUESTION 3

3 Equipartitions **5 / 5**

✓ **+ 5 pts** **Correct**

+ **1 pts** Two examples of equipartitions for arbitrary sets with at least two elements.

+ **1 pts** Clear explanations in the first part

+ **0.75 pts** Explanation of examples is almost clear but definitions are not clearly stated.

+ **0.5 pts** Explanation of examples present but is not thorough or otherwise incorrect

+ **1 pts** Correctly identifying partition sizes for equipartitions of sets of nine elements

+ **1 pts** Correct counting of equipartitions

+ **1 pts** Clear explanation of the second part

+ **0.75 pts** Explanation of the second part is understandable but skips essential details or does not use enough words to explain the ideas. Well articulated but incorrect explanations can be awarded this score.

+ **0.5 pts** Explanation of the second part is included but is not thorough or is otherwise incorrect

+ **0.5 pts** Incorrect counting argument but with significant correct steps.

+ **0 pts** No content

QUESTION 4

4 ALIVE **5 / 5**

✓ **- 0 pts** **Correct**

- 2 pts 5! total arrangements (by multiplication principle)

- 1 pts 2 "EVIL" arrangements (AEVIL and EVILA)

- 2 pts 5! - 2 = 118 arrangements (inclusion/exclusion)

QUESTION 5

5 Fully splitting tree **5 / 5**

✓ **+ 5 pts** **Correct**

+ 1 pts Base case or argument accounts for the tree with one vertex

+ 1 pts Decomposing fully splitting trees in the induction step

+ 0.5 pts Builds a fully splitting binary tree in the induction step (but does not address whether or why they are all built this way)

+ 1 pts Correct use of induction hypothesis about fully splitting binary trees

+ 1 pts Counting arguments have accurate numbers and manipulation of equations and/or inequalities

+ 1 pts Clarity

QUESTION 6

6 Solution counting **5 / 5**

✓ **- 0 pts** **Correct**

- 1 pts Use generalized combination (stars and bars)

- 1 pts Start with 2 "stars" in $$x_1$$

- 1 pts Use 3 bars and the remaining 15 stars to get C(18,3) = 816

- 2 pts Remove the 2 bad solutions (2,15,0,0) and (2,0,15,0) to get 814

- 0 pts Click here to replace this description.

QUESTION 7

7 Minimal spanning tree **5 / 5**

✓ **- 0 pts** **Correct**

- 1.5 pts Incorrect degree for (0,0)

- 1.5 pts Incorrect degree for (2,2)

- 2 pts Incorrect weight

QUESTION 8

8 Bipartite **5 / 5**

✓ **- 0 pts** **Correct**

- 2 pts Assumes the graph is *complete* bipartite without justification

- 2 pts Assumes the bipartition has two sets with specific sizes

- 0 pts Click here to replace this description.

Problem 1. Suppose X and Y are sets with $|X| = 8$ and $|Y| = 2$.

1. How many functions are there from X to Y?
2. How many functions are there from X to Y that are neither injective nor surjective? That is, how many functions are there from X to Y that have the property of being not injective and the property of being not surjective?

1. Each element of X has 2 choices in where it is mapped to in Y. So there are $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^8 = 256$ total functions from X to Y.

2. Since $|X| > |Y|$, there exists no injective functions from X to Y, since it is impossible for 8 inputs to each map to a unique output if the cardinality of the codomain is 2, strictly less than 8. This means we simply have to find how many functions from X to Y are not surjective. Since $|Y| = 2$, this implies all elements of X map to a single element of Y for the function to not be onto. This means there are only <u>two</u> functions from X to Y that are neither surjective or injective. Namely, the function that takes every element of X to the first element of Y, and the function that takes every element of Y to the second element of Y.

1 Counting functions **5 / 5**

- ✓ **+ 2 pts** **a: $$ 2^8 = 256 $$ options total**
- ✓ **+ 3 pts** **b: $$ 2 $$ functions total, by casing on which element of $$ Y $$ is not in the image**
- **+ 0 pts** a: Incorrect count
- **+ 2 pts** b: Correct count, but did not properly justify some steps
- **+ 1.5 pts** b: Incorrect count, but some promising progress.
- **+ 0 pts** b: Incorrect count, as well as several other incorrect statements along the way

Problem 2. Suppose $f : X \to X$ is a function that satisfies

$$f(f(x)) = x$$

for all $x \in X$. Show that f is a bijection.

i) f is injective:

Suppose $f(a) = f(b)$ for some arbitrary $a, b \in X$

Then, $f(f(a)) = f(f(b))$

So $a = b$ (by definition of f)

Since a, b were arbitrary, this shows that $f(a) = f(b)$ implies $a = b$, for all $a, b \in X$, so f is injective.

ii) f is surjective:

proof:

Consider $y \in X$, then $f(y) = x \in X$

So then $f(x) = f(f(y)) = y$, so f is surjective,

since for any $y \in X$, there exists an $x \in X$ such that $f(x) = y$.

Since f is both injective and surjective, f is a bijection (on $X \to X$)

2 f squared equals the identity 5 / 5

✓ + **2 pts** **Injectivity: Let $$ x, y \in X $$ be such that $$ f(x) = f(y) $$. Then $$ f(f(x)) = f(f(y)) $$. Thus $$ x = f(f(x)) = f(f(y)) = y $$, so that $$ x = y $$. Hence, for any $$ x, y \in X $$, if $$ f(x) = f(y) $$, we have $$ x = y $$. We conclude $$ f $$ is injective, as desired.**

+ **1 pts** Injectivity: Partial progress, but informal or missing some important details

+ **0 pts** Injectivity: No serious progress. Arguments that may fall into this category include: trying to use $$ f^{-1} $$ without justifying $$ f $$ is invertible, trying to claim $$ f $$ is the identity function, trying to use the cardinality of $$ X $$ (e.g. assuming $$ |X|< \infty $$), showing injectivity of $$ f \circ f $$ instead of $$ f $$

✓ + **3 pts** **Surjectivity: Let $$ y \in X $$ be arbitrary. Notice $$ f(f(y)) = y $$. Thus, there exists an $$ x \in X $$, namely $$ x = f(y) $$, such that $$ f(x) = y $$. Since $$ y \in X $$ was arbitrary, we see that for all $$ y \in X $$, there exists an $$ x \in X $$ such that $$ f(x) = y $$. Thus, $$ f $$ is surjective, as desired.**

+ **1.5 pts** Surjectivity: Incorrect or missing quantifiers, but argument is correct or almost correct. An informal but (almost) correct argument may also fall into this category.

+ **0 pts** Surjectivity: No serious progress. Arguments that may fall into this category include: trying to use $$ f^{-1} $$ without justifying $$ f $$ is invertible, trying to claim $$ f $$ is the identity function, trying to use the cardinality of $$ X $$ (e.g. assuming $$ |X|< \infty $$), showing surjectivity of $$ f \circ f $$ instead of $$ f $$

Problem 3. If X is a set, an *equipartition* of X is an equivalence relation on X such that every equivalence class has the same size. In other words, E is an equivalence relation on X if $|[x]_E| = |[y]_E|$ for all $x, y \in X$, where $[x]_E = \{z \in X : (x, z) \in E\}$.

1. Show that if $|X| \geq 2$, then there are at least two different equipartitions of X.
2. Suppose $|X| = 9$. How many equipartitions of X are there? *Hint*: First, figure out the possibilities for the cardinality of the equivalence classes. Then count the number of ways of selecting the elements of the classes.

1) For Any set X where $|X| \geq 2$, we can at least find two equipartitions on X:

i) For any set X where $|X| > 0$ applying the equality relation yields $E = \{(x_1, x_1), (x_2, x_2) \ldots (x_n, x_n)\}$ for $x_1 \ldots x_n \in X$ and $|X| = n$. The equivilance class of any $x \in X$ is then $[x]_E = \{x\}$ with $|[x]_E| = 1$, this holds for $|X| > 0$, so it then holds for $|X| \geq 2$, giving a equipartition on X for $|X| \geq 2$.

ii) For any set X where $|X| \geq 2$, we can take the equivilance relation where each element has an ordered pair with every element of X (including itself). i.e. $x, y \in X \Rightarrow (x, y) \in E$ The equivilance class for each $x \in X$ would then contain every element of X, so then $\forall x \in X$, $|[x]_E| = |X|$, so we have another equipartition on X for $|X| \geq 2$.

2) The definition of equipartitions implies that the only candidate cardinalities are $1 \leq |[x]_E| \leq 9$ $\forall x \in X$ if $|X| = 9$. From 1), we know that 1 and 9 are possible values for $|[x]_E|$ $\forall x \in X$, so we must now analyze the possibility of $1 < |[x]_E| < 9$. If $|[x]_E|$ is not either 1 or 9, this means that each $x \in X$ must be related to some, but not all elements of X. The only way to create an equipartition in this situation while still obeying reflexivity, symmetry and transitivity is for the number to be a factor of $|X|$ (i.e. 3 in this case).

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3.21 continued:

Then, the possilities of cardinality of the equivilance classes are 1,3 or 9. 1 and 9 each only have 1 distinct way of selecting elements in the class. The ways to select the elements of X into equivilance classes of cardinality $= 3$ is equiviant to answering the question "How many distinct ways are there to select 3 groups of 3 objects each, from 9 distinct objects?" This is given by $\frac{\binom{9}{3}\times\binom{6}{3}\times\binom{3}{3}}{3!}$, since ordering doesn't matter, so then the total number of equipartitions for $|X| = 9$ is $2 + \frac{\binom{9}{3}\times\binom{6}{3}\times\binom{3}{3}}{3!} = \boxed{282}$

3 Equipartitions 5 / 5

✓ + **5 pts** **Correct**

+ **1 pts** Two examples of equipartitions for arbitrary sets with at least two elements.
+ **1 pts** Clear explanations in the first part
+ **0.75 pts** Explanation of examples is almost clear but definitions are not clearly stated.
+ **0.5 pts** Explanation of examples present but is not thorough or otherwise incorrect
+ **1 pts** Correctly identifying partition sizes for equipartitions of sets of nine elements
+ **1 pts** Correct counting of equipartitions
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+ **0.5 pts** Explanation of the second part is included but is not thorough or is otherwise incorrect
+ **0.5 pts** Incorrect counting argument but with significant correct steps.
+ **0 pts** No content

Problem 4. How many ways are there to arrange the letters ALIVE in such a way that the word EVIL is not contained in the resulting word (as a *consecutive* string of letters)?

total ways to arrangle ALIVE = $5!$

total ways to arrange ALIVE that contains EVIL = 2:

① EVIL A

② A EVIL

so required total $= 5! - 2$

$= \boxed{118}$

4 ALIVE **5 / 5**

- ✓ **- 0 pts** **Correct**
- **- 2 pts** 5! total arrangements (by multiplication principle)
- **- 1 pts** 2 "EVIL" arrangements (AEVIL and EVILA)
- **- 2 pts** 5! - 2 = 118 arrangements (inclusion/exclusion)

Problem 5. Say that a full binary tree is a *fully splitting binary tree* if every non-terminal node has exactly 2 successors and all terminal nodes are at the same level—that is, the length of the unique simple path from the root to a terminal vertex is always the same. Prove that, for all non-negative integers n, if T is a fully splitting binary tree of height n, then T has $2^{n+1} - 1$ vertices.

Proving by induction:
check true for base case of $n=0$:
· if $n=0$, the tree should be a lone root (1 vertex).
Checking: $2^{0+1}-1 = 2-1 = 1$ vertex ✓.

Inductive Step
assume true for $n=k$, i.e. for height k, T has $2^{k+1}-1$ vertices (inductive hypothesis). We want to show for height $k+1$, T has $2^{(k+1)+1}-1 = 2^{k+2}-1$ vertices. Note that for a full splitting tree with height $n+1$, the internal vertices of this tree comprise the full splitting tree with height n. Then, using our inductive hypothesis, if T has height $k+1$, it has $2^{k+1}-1$ internal vertices. We know that any full binary tree with i internal vertices has $2i+1$ total vertices. So then if T has height $k+1$, it has $2[2^{k+1}-1]+1$ total vertices.
Note: $2[2^{k+1}-1]+1 = 2^{k+2}-1$ vertices, as required.

So we know the statement is true for $n=0$ and if the statement holds for $n=k$, it holds for $n=k+1$. So, by induction, the statement holds $\forall\, n \geq 0$

i.e. A fully splitting tree with height $n \geq 0$ has $2^{n+1}-1$ total vertices.

5 Fully splitting tree 5 / 5

✓ + **5 pts** **Correct**

+ **1 pts** Base case or argument accounts for the tree with one vertex
+ **1 pts** Decomposing fully splitting trees in the induction step
+ **0.5 pts** Builds a fully splitting binary tree in the induction step (but does not address whether or why they are all built this way)
+ **1 pts** Correct use of induction hypothesis about fully splitting binary trees
+ **1 pts** Counting arguments have accurate numbers and manipulation of equations and/or inequalities
+ **1 pts** Clarity

Problem 6. How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 = 17$$

where x_1, x_2, x_3, and x_4 are non-negative integers satisfying $x_1 \geq 2$, $x_2 \leq 14$, and $x_3 \leq 14$.

- $x_1 \geqslant 2$
- $x_2 \leq 14$
- $x_3 \leq 14$
- $x_4 \geqslant 0$

let $x_1' = x_1 - 2$, then we need to solve

$$x_1' + x_2 + x_3 + x_4 = 15$$

for $x_2 \leq 14, x_3 \leq 14$ ($x_1' \geqslant 0, x_4 \geqslant 0$ implied)

- bad solutions for $x_2 = 15$ or $x_3 = 15$

$\Rightarrow$ 2 bad solutions ($x_1 = x_2 = x_4 = 0,\ x_3 = 15$ OR $x_1 = x_3 = x_4 = 0,\ x_2 = 15$)

Using stars and bars, the total number of solutions is

$$\binom{n+k-1}{k-1} - 2, \quad \text{for } n = 15, k = 4$$

i.e.

$$\text{total} = \binom{15+4-1}{4-1} - 2 = \binom{18}{3} - 2$$

$$= \boxed{814} \text{ solutions}$$

6 Solution counting **5 / 5**

- ✓ **- 0 pts** **Correct**
- **- 1 pts** Use generalized combination (stars and bars)
- **- 1 pts** Start with 2 "stars" in $$x_1$$
- **- 1 pts** Use 3 bars and the remaining 15 stars to get C(18,3) = 816
- **- 2 pts** Remove the 2 bad solutions (2,15,0,0) and (2,0,15,0) to get 814
- **- 0 pts** Click here to replace this description.

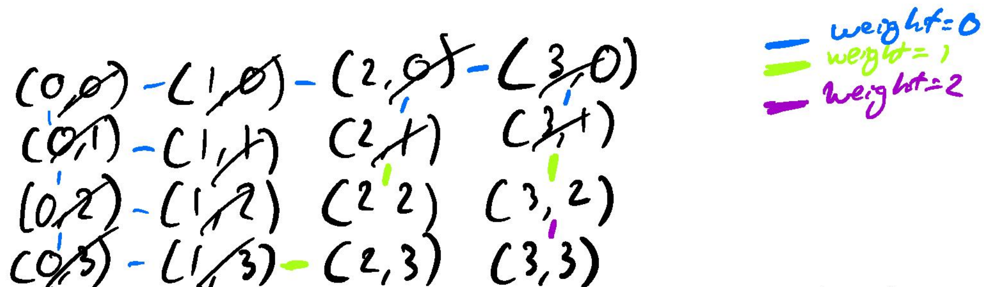
Problem 7. Let G be a graph whose vertices consist of pairs (a, b) where a and b are non-negative integers less than or equal to 3. Say that there is an edge between (a, b) and (c, d) if either $|a - c| = 1$ and $b = d$ or $|b - d| = 1$ and $a = c$, where $|x|$ denotes the absolute value of x. In other words, two vertices are connected by an edge if they differ by exactly 1 in one coordinate and agree on the other.

1. What is the degree of $(0, 0)$? 2
2. What is the degree of $(2, 2)$? 4
3. Define the weight of an edge between (a, b) and (c, d) to be the minimum of the numbers a, b, c, d. What is the weight of a minimal spanning tree for the graph G? 5

1] possible edges from (0,0) are (1,0) or (0,1), so (0,0) has degree 2 [(a,b) are non-negative]

2] possible edges from (2,2) are (2,3), (3,2), (1,2) or (2,1), so (2,2) has degree 4

3] MST: *All vertices represented as ordered pairs



Note: · Any vertex with a 0 or a 1 can be connected to an edge with weight 0, since every vertex with a 1 is connected with a vertex containing 0.

· Any vertex with nothing less than 2 can be connected with an edge with weight 1

· Any vertex with nothing less than 3 can be connected with an edge with weight 2.

So then, the minimal spanning tree will have weight = 1+1+1+2 = 5 (as shown above)

7 Minimal spanning tree 5 / 5

- ✓ **- 0 pts** **Correct**
- **- 1.5 pts** Incorrect degree for (0,0)
- **- 1.5 pts** Incorrect degree for (2,2)
- **- 2 pts** Incorrect weight

Problem 8. Suppose G is a graph with 17 vertices. Show that if G has at least 9 vertices of degree 9, then G cannot be bipartite.

Assume G is bipartite with parts U, V and has 17 vertices. If $u \in U$ has degree 9, that means V has at least 9 vertices, by definition of a bipartite graph, since u can only form edges with vertices in V. Since there are at least 9 vertices in V, there is no more than 8 vertices in U, since G has 17 total vertices. We now require at least 8 more vertices to have degree 9. These remaining vertices cannot be in V, since vertices in V can have a maximum of degree 8, since they only can form edges with maximum 8 vertices in U. So then the remaining vertices with degree 9 must be in U. However, U only has at most 7 more vertices (other than u), so we face a contradiction. So, for G to maintain 17 vertices and have at least 9 vertices of degree 9, G cannot be bipartite.

8 Bipartite 5 / 5

✓ **- 0 pts** **Correct**

- 2 pts Assumes the graph is *complete* bipartite without justification

- 2 pts Assumes the bipartition has two sets with specific sizes

- 0 pts Click here to replace this description.