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MATH 61 - FINAL EXAM

0.1. **Instructions.** This is a 180 minute exam. You should feel free to quote any theorems proved in class, as well as anything proved in the homework or discussion section. There are 12 questions—on the exam, you are required to do the first true/false question, and choose 9 of the remaining 11. Only 9 problems other than the true/false question will be graded so *you should indicate which problems you want graded by marking the one you do not want graded with an X*, in the case that you attempt all 6. Each question is worth 10 points. Unless otherwise specified, you are required to justify your answers.

Recall all of our graphs are simple.

Exercise 0.1. Indicate whether the following statements are true or false:

(1) The number of rooted trees on a fixed set of n vertices is n^{n-2}.
(2) Suppose $|X| = n$ and $|Y| = k$. There are n^k many functions from X to Y.
(3) If G has no subgraph isomorphic to $K_{3,3}$ or K_5, then G is planar.
(4) If $G = (V, E)$ is a graph, then the relation R on V, defined by $(x, y) \in R$ if and only if there is a path from x to y in G, is an equivalence relation.
(5) If $T = (V, E)$ is a rooted tree, then the relation D on V, defined $(x, y) \in D$ if x is a descendent of y or if $x = y$, is a partial order.

(1) F number of trees, not rooted ✓
(2) F k^n ✓
(3) F homeomorphic ✓
(4) T
(5) T

Date: December 9, 2018; Ramsey.

Exercise 0.2. Recall that for X a set, $\mathcal{P}(X)$ denotes the *power set* of X, the set whose elements are the subsets of X: $\mathcal{P}(X) = \{Y : Y \subseteq X\}$. Show that ✓

$$\sum_{i=1}^{n} |\mathcal{P}(\{1,\ldots,i\})| = 2^{n+1} - 2.$$

$P(\{1,\ldots,i\})$ is the power set of a set with i elements.

The number of subsets is 2^i

$$\sum_{i=1}^{n} |P(\{1,\ldots,i\})| = \sum_{i=1}^{n} 2^i$$

$$= 2^1 + 2^2 + \cdots + 2^n = 2(2^0 + 2^1 + \cdots + 2^{n-1})$$

$$= 2\left(\frac{2^n - 1}{2-1}\right) = 2(2^n - 1)$$

$$= \boxed{2^{n+1} - 2}$$

All steps are reversible.

(*) Proof of $\sum_{i=0}^{n-1} 2^i = \frac{2^n - 1}{2-1}$

$2^n - 1 = (2-1)(2^{n-1} + 2^{n-2} + \cdots + 1)$ By factorization

Or by induction: base case: $n = 1$: $2^0 = \frac{2^1 - 1}{2-1} = 1$ ✓

Assume $\sum_{i=0}^{k-1} 2^i = \frac{2^k - 1}{2-1}$

Then $\sum_{i=0}^{k} 2^i = \sum_{i=0}^{k-1} 2^i + 2^k = 2^k - 1 + 2^k = 2^{k+1} - 1$. □

Exercise 0.3. Show that, for all natural numbers n,

$$2^n = \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = \sum_{i=0}^{n} \binom{n}{i}.$$

✓

Note: Both algebraic and combinatorial proofs are possible.

Combinatoric:

Let S be a set $\{s_1, s_2, \ldots, s_n\}$ of size n.

A subset has some number of these elements.

Consider each element: either they are in ~~the~~ a given subset or not.

$\Rightarrow$ 2 choices

Thus, we have $2 \cdot 2 \cdot 2 \cdots 2 = 2^n$ choices for a subset.

If we recount with casework: there is $\binom{n}{i}$ ways of picking a subset with i elements.

$\Rightarrow$ sum through case where $i = 0$ to $i = n$

$\Rightarrow \sum_{i=0}^{n} \binom{n}{i}$. ∎

Algebraic:

Binomial Theorem of $(1+1)^n$

is $\binom{n}{0} 1^n \cdot 1^0 + \binom{n}{1} 1^{n-1} \cdot 1^1 + \ldots + \binom{n}{n} 1^0 \cdot 1^n$

$\Leftrightarrow (1+1)^n = \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n}$

$2^n = \sum_{i=0}^{n} \binom{n}{i}$. ∎

Exercise 0.4. There are $\binom{52}{5}$ many 5-card hands from a standard 52 card deck. How many of them are either a *straight* or a *flush*? (A *straight* is a sequence of 5 cards whose values are in order, the ace is high (and not low); a *flush* is 5 cards of the same suit).

√

Straight: 2, 3, 4, 5, 6 to 10, J, Q, K, A

9 of these

For each one, we have $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$ choices for the suits.

$\Rightarrow 4^5 \cdot 9$

Flush: 4 choices for suit

Then $\binom{13}{5}$ choices for the 5 cards in this suit

$\Rightarrow 4\binom{13}{5}$

Both Straight and Flush:

4 choices for suits

9 choices for the flush (2, 3, 4, 5, 6 to 10, J, Q, K, A)

$= 4 \cdot 9$

By Principle of I-E: # straight or flush is

$$= \boxed{4^5 \cdot 9 + 4\binom{13}{5} - 4 \cdot 9}$$

Exercise 0.5. How many integer solutions are there to the equation

$$x_1 + x_2 + x_3 = 37$$

subject to the constraints that $3 \le x_1 < 6$, $4 \le x_2$, and $0 \le x_3 < 37$.

√

We have 3 choices for x_1 : 3, 4, 5

We have ~~37~~ 34 $- x_1$ choices for x_2 : 4, 5, … $37 - x_1$

x_3 is determined by x_1 and x_2

x_3 cannot ever be 37 so we ignore that

$$\Rightarrow \sum_{x_1=3}^{5} 34 - x_1 = 34 \cdot 3 - \sum x_1 = 162 - 12 = \boxed{90}$$

Alternate: PIE and sticks and stones

$x_1' = x_1 - 3$ $\quad x_2' = x_2 - 4$ $\quad x_3' = x_3$

$0 \le x_1' < 3$ $\quad x_2 \ge 0$ $\quad 0 \le x_3$ ~~< 37~~ since sum is 30

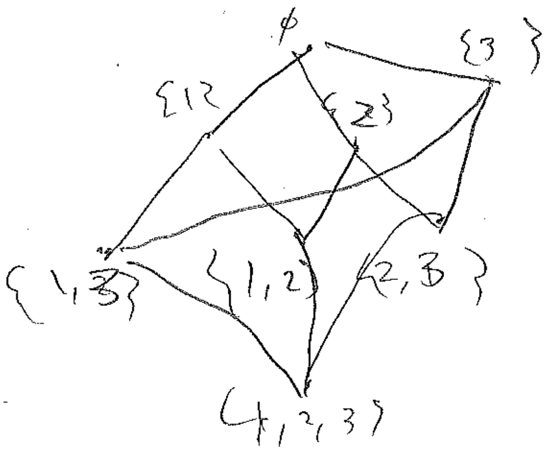
~~$\binom{39}{2}$~~ –

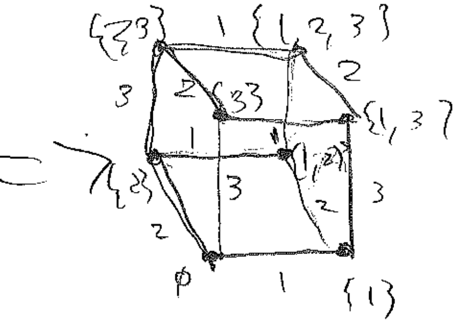
$x_1' + x_2' + x_3' = 30$

$$\binom{32}{2} - \binom{29}{2} = \frac{32 \cdot 3\cancel{0} - 29 \cdot 28}{2} = \frac{29 \cdot 3 + 28 \cdot 3 + 9}{2} = \frac{60 \cdot 3}{2} = \boxed{90}$$ √

Exercise 0.6. Consider a weighted graph $G = (V, E)$ where $V = \mathcal{P}(\{1,2,3\})$ and v and w are connected by an edge if and only if $|v \Delta w| = 1$, in which case this edge is given weight corresonding to the unique element of $v \Delta w$. In other words, the vertices of G are the subsets of $\{1,2,3\}$ and the edges are pairs of sets where one has exactly one more element than the other. For example, $\{1\}$ and $\{1,2\}$ are connected by an edge, and the weight of this edge is 2.

(1) Is there some d so that G is d-regular?
(2) What is the sum of the weights in a minimal spanning tree?





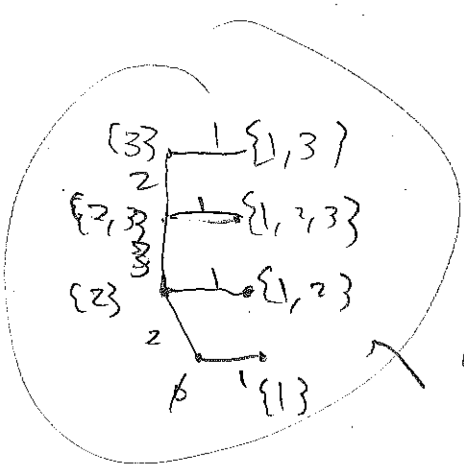
Yes:

(1) $d = 3$ each vertex connects to 3 other vertex

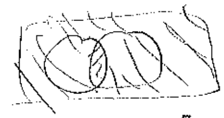
(2) $1+2+1 + 1+2+1+3$

$= 11$

for bottom layer:
$1+2+1$
for top layer:
$1+2+1$
for bottom to top:
3



One example of a MST



Exercise 0.7. Suppose X is the universal set and for subsets $A \subseteq X$, we write $\overline{A}$ to denote the complement of A in X. The *Sheffer stroke* $\uparrow$ is a single operation on sets defined by $A \uparrow B$ is the complement of $A \cap B$. This has the remarkable property that every other operation on sets may be defined in terms of it.

(1) Show $\overline{A} = (A \uparrow A)$.
(2) Show $A \cup B = (A \uparrow A) \uparrow (B \uparrow B)$
(3) Find a formula for $A \cap B$ using only parentheses and $\uparrow$.

(1) $A \uparrow A$ is the complement of $A \cap A$

$\Leftrightarrow \overline{A \cap A} = \overline{A}$ □.

(2) $A \uparrow A = \overline{A}$ by (1)

$B \uparrow B = \overline{B}$ by 2

Complement of a set $S \subseteq X$ is $X - S$ by definition.

$\overline{A} \uparrow \overline{B} = \overline{\overline{A} \cap \overline{B}} = X - \overline{A} \cap \overline{B}$

$= X - (X - A) \cap (X - B)$

$= X - (X \cap X - A \cap X - B \cap X + A \cap B)$

$= X - (X - A - B + A \cap B)$

$= A + B - A \cap B = A \cup B$ by Principle of Inclusion Exclusion □

(3) $A \cap B = A + B - A \cup B$ By (2)

By (1) $= A \uparrow A + B \uparrow B - (A \uparrow A) \uparrow (B \uparrow B)$

Exercise 0.8. At the end of finals week there will be a party with 15 guests.

(1) If each person shakes every other person's hand exactly once, how many handshakes have taken place?
(2) Suppose that 7 of the guests have the same cold, so they don't want to shake the hands of the guests who are not sick, but they don't mind shaking each others' hands. If all the sick guests shake hands and all the not-sick guests shake hands, how many hand shakes have taken place?

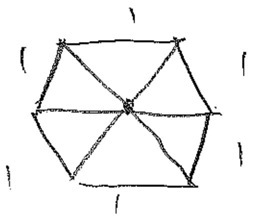
(1) $\binom{15}{2} = \frac{15 \cdot 14}{2} = \boxed{105}$ choose two people for a handshake

(2) 7 sick 8 not sick

$\binom{7}{2} + \binom{8}{2} = \frac{7 \cdot 6}{2} + \frac{8 \cdot 7}{2} = \boxed{49}$ handshakes

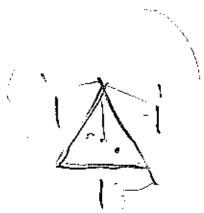
Exercise 0.9. Suppose there are 7 points in the interior of a regular hexagon (i.e. all interior angles are the same) of side length 1. Show that at least two of the points are within distance $\frac{1}{2}$ of one another.

reg. hexagon :



If we split the hexagon in the 6 equilateral triangle regions with sidelength 1,

2 points must be within the same region by Pigeonhole.

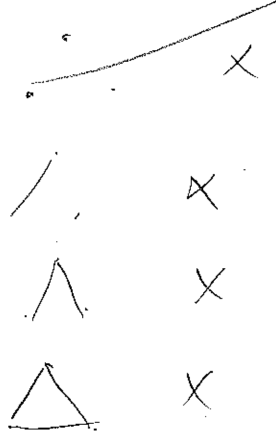


If we draw the circle centered at one vertex with radius 1, it passes through only the other two vertices,

so the maximum distance is 1.

$\Rightarrow$ 2 points are within 1 of each other.

Exercise 0.10. Recall that a graph G is called *rigid* if the only autorphism of G is the identity function - i.e. the function that sends $v \mapsto v$ and $e \mapsto e$ for every vertex and edge of G respectively. Is there a rigid graph with exactly 3 vertices? Give an example or show there is no such graph.



No graph.

Exercise 0.11. Suppose X is a set with 30 elements and Y is a set with 20 elements. How many functions $f : X \to Y$ are there satisfying both of the following properties (at the same time):

(1) $f : X \to Y$ is onto (i.e. surjective)
(2) For every $y \in Y$, $|\{x \in X : f(x) = y\}| \le 2$.

30 to 20

=> for the codomain, let x_i be the number of mappings to the ith element!

=> $x_1 + x_2 + \dots + x_{20} = 30$

$2 \ge x_1 \ge 1, 2 \ge x_2 \ge 1, \dots, 2 \ge x_{20} \ge 1$

$x'_i = x_i - 1$

$1 \ge x'_1 \ge 0, 1 \ge x'_2 \ge 0, \dots, 1 \ge x'_{20} \ge 0$

We have 20! ways of picking the surjective part

=> $x'_1 + x'_2 + \dots + x'_{20} = 10$

~~sticks and stones: $\binom{29}{19}$~~

Either 0 or 1 => we want to pick 10 x_i' such that they all are 1, the rest are 0:

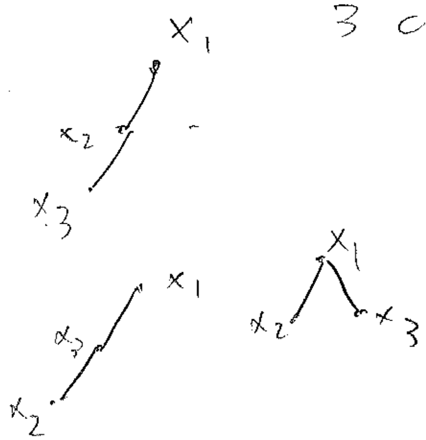
=> ~~$\binom{20}{10}$ subsets~~ => $\binom{20}{10}$ subsets

$$\boxed{\binom{20}{10} \cdot 20!}$$

Alternate.

For the first 20 elements of domain, there are 20! ways of

Exercise 0.12. How many rooted trees are there on the vertices $\{x_1, x_2, x_3\}$? Are there more or fewer than the total number of graphs on these vertices?



3 choices for root, let it be x_1 WLOG.

$3 \cdot 3 = \boxed{9}$ or 3^{3-1} trees

$\cdot$ 3 for root

$= 3^2 = \boxed{9 \text{ rooted trees}}$

total number of graphs: $2^{\binom{3}{2}} = \boxed{8}$

More rooted trees than total graphs. $9 > 8$. ∎