# Midterm 2 Solutions

1. (a) (5 points) Show that for $n \in \mathbb{N}$, and $1 \leq k \leq n$ that

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}.$$

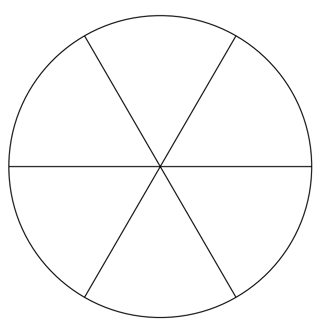
See the proof in the book (Theorem 6.7.6).

(b) (5 points) Prove that for $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

See the proof in the book (Theorem 6.7.1).

2. Suppose there is a circle with six sectors:



If you place 31 points inside the circle, are the following statements true or false (prove your answer)?

(a) (3 points) There must be a sector with at least 6 points.

This is true by the pigeonhole principle. There are 6 holes (sectors) and 31 pigeons (points). So there must be at least $\lceil \frac{31}{6} \rceil = 6$ points in one sector.

(b) (3 points) There must be a sector with at most 4 points.

This is false. Places 5 points in 5 sectors and 6 points in the last sector.

(c) (4 points) There must be two neighboring sectors which contain 11 points.

This is true. Separate the sectors into 3 neighboring pairs. Then there are 3 holes (neighboring pairs) and 31 pigeons (points). So there must be at least $\lceil \frac{31}{3} \rceil = 11$ points in one neighboring pair of sectors.

3. (8 points) Let $x_n$ be defined by the recursion,

$$x_n = x_{n-1} + 3x_{n-2} + 56x_{n-3},$$

where $x_0 = 1, x_1 = 4, x_2 = 10$. Prove that

$$x_n < 3 \cdot 5^n,$$

for all $n \in \mathbb{N}$.

Prove this by induction. First prove the base case with all the initial conditions:

$$x_0 = 1 < 3, \quad x_1 = 4 < 3 \cdot 5, \quad x_2 = 10 < 3 \cdot 25.$$

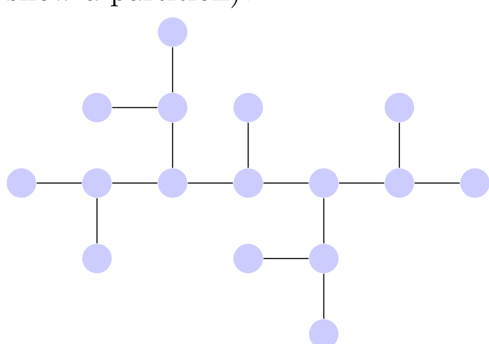
Inductive step:

$$x_n = x_{n-1} + 3x_{n-2} + 56x_{n-3} < 3 \cdot 5^{n-1} + 9 \cdot 5^{n-2} + 168 \cdot 5^{n-3},$$

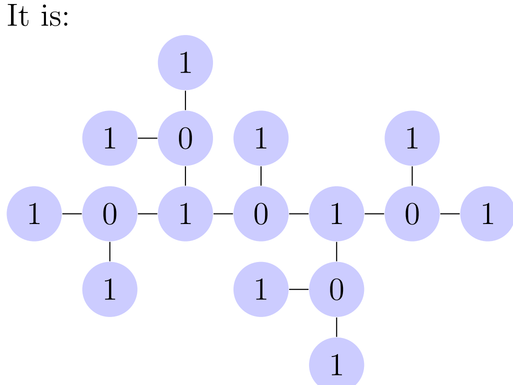
by the induction hypothesis. So

$$x_n < \frac{3}{5} \cdot 5^n + \frac{9}{25} \cdot 5^n + \frac{168}{125} 5^n$$
$$< \frac{75 + 45 + 168}{125} 5^n = \frac{288}{125} 5^n < 3 \cdot 5^n.$$

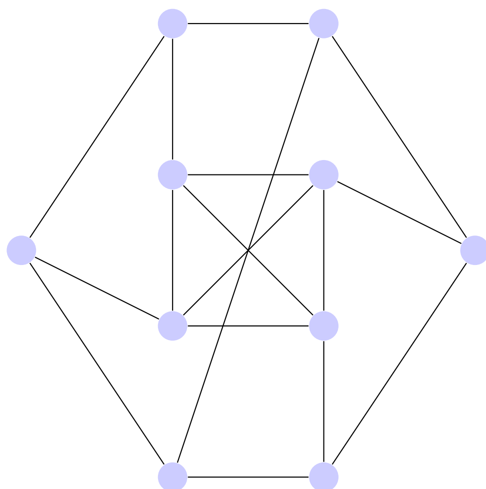
4. (a) (5 points) Is the following graph bipartite (either prove that it is not or show a partition)?



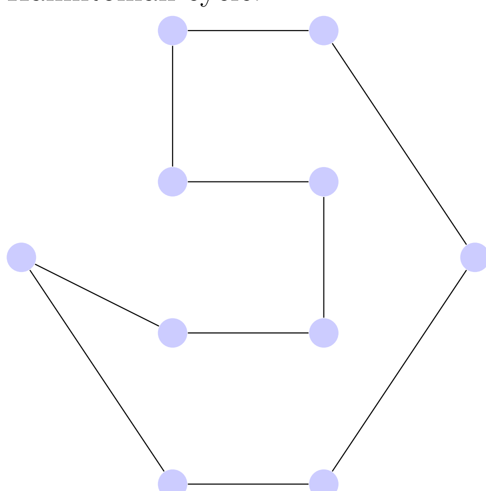
It is:



(b) (3 points) Determine whether the following graph contains a Hamiltonian cycle.



Hamiltonian cycle:



(c) (2 points) Determine whether the graph from part (b) contains an Euler cycle.

It does not since there are vertices with odd degree.

5. (7 points) Solve the following recursion relation:

$$a_n = 2^n (a_{n-1})^5 (a_{n-2})^{-6},$$

where $a_0 = 2^{15/4}$ and $a_1 = 2^{29/4}$.

By taking the base 2 logarithm of both sides, we get a new recurrence:

$$b_n = 5b_{n-1} - 6b_{n-2} + n,$$

with $b_0 = 15/4$ and $b_1 = 29/4$, where $b_n = \log_2 a_n$. This is an inhomogeneous linear recurrence. The homogeneous solution is:

$$b_n^h = C_1 2^n + C_2 3^n.$$

A guess for $b_n^p$ is $k_1 n + k_2$. Solving for $k_1$ and $k_2$ in the recurrence gives that

$$b_n^p = \frac{1}{2}n + \frac{7}{4}.$$

So

$$b_n = C_1 2^n + C_2 3^n + \frac{1}{2}n + \frac{7}{4}.$$

We use initial conditions to solve for $C_1 = C_2 = 1$. So

$$b_n = 2^n + 3^n + \frac{1}{2}n + \frac{7}{4}$$

and

$$a_n = 2^{2^n + 3^n + \frac{1}{2}n + \frac{7}{4}}.$$