Midterm 1 Solutions

1. (5 points) If a and b are in $\mathbb{Z}$, then we say that a is divisible by b if there exists an integer k with $a = kb$. Show $5^n - 1$ is divisible by 4 for all $n \geq 1$.

   Induction: For $n = 1$, $5^n - 1 = 4$ which is divisible by 4.

   n implies $n + 1$:

   $$5^{n-1} - 1 = 5^n(5) - 1 = (5^n - 1) + 5^n(4).$$

   By the induction hypothesis, there exists an integer k so that

   $$(5^n - 1) + 5^n(4) = (4)k + 5^n(4) = 4(k + 5^n).$$

   So $5^{n+1} - 1$ is divisible by 4.

2. (5 points) Show that $n! \geq 2^{n-1}$ for all $n \geq 1$.

   Induction: For $n = 1$, both sides of the inequality are 1,

   n implies $n + 1$:

   $$(n + 1)! = (n + 1)n! \geq (n + 1)2^{n-1}$$

   by the induction hypothesis. Since $n \geq 1$,

   $$(n + 1)2^{n-1} \geq (2)2^{n-1} = 2^n.$$

   So $(n + 1)! \geq 2^n$.

3. (5 points) Let $A \subset X$. Show that $X \setminus (X \setminus A) = A$.

$$\begin{aligned} x \in X \setminus (X \setminus A) \Leftrightarrow \\ x \notin X \setminus A \Leftrightarrow \\ x \in A \end{aligned}$$

4. (5 points) Define $f \colon \mathbb{Z} \to \{0, 1\}$ as

$$f(n) = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Is f one-to-one? Is f onto? (prove your answer)

$f(0) = f(2) = 1$, so f is not one-to-one.

$f(0) = 1$ and $f(1) = 0$, so f is onto.

5. (a) (5 points) Let $f \colon X \to Y$. Define a relation on X, $x_1 \sim x_2$ if $f(x_1) = f(x_2)$. Show that this is an equivalence relation.

   To show that this relation is an equivalence relation, we need to show that it is reflexive, symmetric and transitive.
   Reflexive: Let $x \in X$, then $f(x) = f(x)$. So $x \sim x$.
   Symmetric: Let $x_1, x_2 \in X$ and suppose $x_1 \sim x_2$. Then $f(x_1) = f(x_2)$ and therefore $f(x_2) = f(x_1)$. So $x_2 \sim x_1$.
   Transitive: Let $x_1, x_2, x_3 \in X$ and suppose $x_1 \sim x_2$ and $x_2 \sim x_3$. Then $f(x_1) = f(x_2)$ and $f(x_2) = f(x_3)$. So $f(x_1) = f(x_3)$ and $x_1 \sim x_3$.

   (b) (5 points) Using f from Problem 4, describe the equivalence classes on $\mathbb{Z}$ defined by part (a). How many equivalence classes are there?

Using f in problem 4 we see that $n \sim m$ in $\mathbb{Z}$ if either n and m are both even or n and m are both odd. So there are two equivalence classes:

$$[0] = \{n \in \mathbb{Z} : n \text{ is even}\}$$

and

$$[1] = \{n \in \mathbb{Z} : n \text{ is odd}\}.$$

6. (5 points) Let letters A,B,C,D,E,F,G be used to form strings of length 4. How many strings of length 4 with repetitions contain A and B. How about without repetitions?

With repetitions: Let X be the set of strings that contain A and let Y be the set of strings that contain B. We want to compute $X \cap Y$. Using the inclusion exclusion principle,

$$|X \cup Y| = |X| + |Y| - |X \cap Y|.$$

So instead let us compute the other terms in the equation. There are 7^4 total strings and there are 5^4 strings that contain neither A nor B. So

$$|X \cup Y| = 7^4 - 5^4.$$

There are 6^4 strings do not contain A so

$$|X| = 7^4 - 6^4.$$

Similarly,

$$|Y| = 7^4 - 6^4.$$

So

$$|X \cap Y| = (2)(7^4 - 6^4) - (7^4 - 5^4).$$

Without repetitions: Choose two places for A and B, there are 4 choices for where to place A and then 3 choices for where to place B. In the remaining two places there are 5 choices for the first letter and 4 choices for the second letter. So there are

$$(4)(3)(5)(4)$$

total strings.

7. Consider strings of length 8 made up of elements in $\{0,1\}$.

(a) (3 points) How many strings contain 000 as a substring?

We count instead how many strings don't contain 000. Call this S_8. Any string must end with 1,10, or 100. So

$$S_8 = S_7 + S_6 + S_5.$$

This recursion works for shorter length strings as well.

$$\begin{aligned} S_8 &= (S_6 + S_5 + S_4) + S_6 + S_5 = 2S_6 + 2S_5 + S_4 \\ &= 2(S_5 + S_4 + S_3) + 2S_5 + S_4 = 4S_5 + 3S_4 + 2S_3 \\ &= 4(S_4 + S_3 + S_2) + 3S_4 + 2S_3 = 7S_4 + 6S_3 + 4S_2. \end{aligned}$$

We can now compute each of these terms. S_4 is the number of strings without 000. These are everything but:

$$0000, 0001, 1000.$$

So $|S_4| = 2^4 - 3 = 13$. Similarly, $|S_3| = 2^3 - 1 = 7$ and $|S_2| = 2^2 = 4$. So

$$|S_8| = (7)(13) + (6)(7) + (4)(4) = 149.$$

So the answer is $256 - 149 = -107$.

(b) (3 points) How many strings contain three 0's?

$$\binom{8}{3} + \binom{8}{4} + \binom{8}{5} + \binom{8}{6} + \binom{8}{7} + \binom{8}{8}$$

(c) (4 points) How many string contain more 0's than 1's?

$$\binom{8}{5} + \binom{8}{6} + \binom{8}{7} + \binom{8}{8}$$

8. Let $S \subset \mathbb{N}$ be a set. An element $l \in S$ is called the *least element* in S if for all $s \in S$, $l \leq s$. Let also $A_k = \{n \in \mathbb{N} : n \leq k\}$.

(a) (5 points) Prove by induction that A_k either contains l (the least element of S) or $A_k \cap S = \emptyset$.

For $k = 1$: $A_1 = \{1\}$. If $1 \in S$, then it is the least element since 1 is the least element in $\mathbb{N}$. If $1 \notin S$, then $A_1 \cap S = \emptyset$.

k implies $k+1$: By the induction hypothesis either $A_k \cap S = \emptyset$ or $l \in A_k$. If $l \in A_k$, then $l \in A_{k+1}$.

If $A_k \cap S = \emptyset$ and $A_{k+1} \cap S \neq \emptyset$, then $k+1 \in S$. Suppose there is an $s \in S$ so that $s < k+1$. Then $s \in A_k$ and this would contradict our assumption. So $k+1 = l$ and $l \in A_{k+1}$. Otherwise $A_{k+1} \cap S = \emptyset$.

(b) (5 points) Show that if S is nonempty, then S has a least element (hint: use (a)).

Since S is nonempty there exists an element $k \in S$ and so $k \in A_k$. Therefore $S \cap A_k \neq \emptyset$. By (a), $l \in A_k$ and thus S has a least element.