21F-MATH61-2 Midterm 2

[redacted]

TOTAL POINTS

24 / 25

QUESTION 1

1 Question 1 **5 / 5**

✓ **+ 2 pts Part a**

✓ **+ 3 pts Part b**

+ 0 pts Incorrect

QUESTION 2

2 Question 2 **6 / 7**

+ 7 pts Correct

- 1 pts Left any of your answers in terms of P(n,k) or C(n,k), which the instructions at the beginning of the exam tell you not to

✓ **+ 2 pts a) Correct**

+ 1 pts a) Wrote P(26,7)=26!/7!

+ 0.5 pts a) C(26,7) instead of P(26,7)

✓ **+ 2 pts b) Correct**

- 1 pts b) No work shown

+ 1.5 pts b) Minor error

+ 1 pts b) Partial progress

+ 1 pts b) Only counted those starting (or ending) with "BC" or "CB"

+ 0.5 pts b) A flawed strategy which, if tweaked, might work

+ 0 pts b) Incorrect

+ 3 pts c) Correct

- 0.5 pts c) Minor error

✓ **+ 2 pts c) One of the terms in the inclusion/exclusion is incorrect**

- 1 pts c) Did not do inclusion/exclusion correctly, subtracted the "ABC" and "EFG" term instead on adding it

+ 1.5 pts c) Did not do inclusion/exclusion correctly, dropped one of the terms.

+ 1 pts c) Correctly used inclusion/exclusion and gave some attempt at computing the terms but did not do so correctly.

- 1 pts c) No work shown

+ 0.5 pts c) Correctly used inclusion/exclusion but did not compute any of the terms

+ 0 pts c) Incorrect

(1) $$6\cdot 20$$, not $$6+20$$

QUESTION 3

3 Question 3 **6 / 6**

✓ **- 0 pts (a) Correct.**

✓ **- 0 pts (b) Correct.**

QUESTION 4

4 Question 4 **7 / 7**

✓ **+ 1 pts (a): Correct**

✓ **+ 3 pts (b): Correct**

+ 2 pts (b): Partial solution

+ 0 pts (b): Totally incorrect/no meaningful progress

+ 3 pts (c): Correct

✓ **+ 3 pts (c): Correct method, error in calculation**

+ 2.5 pts (c): Correct method, easy-to-spot error in solution

+ 2.5 pts (c): Correct, but no explicit formula

+ 2 pts (c): Partial solution

+ 1.5 pts (c): Partially correct method

+ 0 pts (c): Incorrect/no meaningful progress

+ 0 pts (a): Incorrect

+ 1 pts (b): Some progress

+ 1 pts (c): Some progress

Midterm 2

Name: [redacted] __________

UID: __________

Instructions: Do not open this exam until instructed to do so. You will have 50 minutes to complete the exam. Please print your name and student ID number above. You may **not** use calculators, books, notes, or any other material to help you. Please make sure your phone is silenced and stowed where you cannot see it. You may use the front and back of the page for your answers. **Do not write answers for one question on the page of another question**. If you need scratch paper, please ask one of the proctors. You must **show all your work** to receive credit.

Note: In this entire exam, you may leave your responses written as products, fractions, and with factorials. However, you are not allowed to leave expressions of the form $C(n, k)$, $P(n, k)$, etc. As always, you need to justify all your answers.

Question	Points	Score
1	5	
2	7	
3	6	
4	7	
Total:	25	

[redacted]

1. Suppose Matt the baker sells 5 types of bread, 7 types of pastries and 3 types of tarts. All his clients buy exactly one bread, one pastry and one tart every time they visit the bakery.

   (a) (2 points) If in one day, 150 people show up at the bakery, show that there are at least 2 people who bought the exact same bread, pastry and tart.

   (b) (3 points) How many people would have to show up to make sure that at least 5 people buy the same bread and pastry?

a) There are 5·7·3 total combinations of one bread, one pastry, and one tart. $5 \cdot 7 \cdot 3 = \begin{array}{r} 35 \\ \times 3 \\ \hline 105 \end{array} = 105.$

Therefore, by the pidgeon hole principle, there must be at least 2 people who bought the same bread, pastry, and tart, as there are 105 possible unique combinations and 150 > 105 people.

b) 5·7 = 35 pidgeonholes. 4 pidgeons/hole = 140 pidgeons needed to fill all of the holes with at least 4 pidgeons. (most optimal distribution).

∴ We need at least 140+1 = 141 people to show up to ensure that there are at least 5 people who buy the same bread and pastry.

$\begin{array}{r} 35 \\ \times 4 \\ \hline 140 \end{array}$

[redacted]

2. (a) (2 points) How many words of 7 letters can we form with the 26 letters of the Latin alphabet if we don't allow letters to be repeated in a word?

   (b) (2 points) How many such words can we form if we require B and C to appear in it next to each other (in either order)?

   (c) (3 points) How many such words (as in part (a)) can we form if we don't allow the strings ABC or EFG to appear in it?

a) 7 letter words with no repeats:

$= 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 = \frac{26!}{(26-7)!} = \frac{26!}{19!}$

↑ 26 choices for first

↑ 25 choices for 2nd (removed 1)

etc...

b) BC _ _ _ _ _

_ BC _ _ _ _ etc.

$= 2 \cdot \binom{6}{1} \cdot P(24, 5) = 2 \cdot 6 \cdot \frac{24!}{19!} = 2 \cdot 6 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20$

↑ 2 ways to order BC or CB

↑ Choose 1 of 6 places to put BC/CB

↑ order the remaining 5 letters

c) Total possible strings $= \frac{26!}{19!}$

Strings containing ABC: ABC _ _ _ _

$= \binom{5}{1} \cdot P(23, 4)$

Strings containing EFG: $= \binom{5}{1} \cdot P(23, 4)$

Strings containing both ABC and EFG:

_ EFG ABC

_ ABC EFG

EFG _ ABC

EFG ABC _

ABC _ EFG

ABC EFG _

Order ABC, EFG: $P(3, 2) = 6$

Choose last letter: 20 choices

= 20 strings with both

∴ # words not allowing "ABC" or "DEF"

$= \frac{26!}{19!} - \left(2\left(5! \cdot \frac{23!}{19!}\right) - 26\right)$

↑ Incl. Excl principle

[redacted]

3. (a) (3 points) How many distinct 12-digit numbers can we form with the digits 1, 1, 4, 4, 4, 4, 5, 5, 5, 8, 8, 9?

   (b) (3 points) In how many of those numbers do the odd digits appear in increasing order? (For instance, 144185584594 is such a number.)

a) #1: 2
#4: 4
#5: 3
#8: 2
#9: 1

$\rightarrow \binom{12}{2}\binom{10}{4}\binom{6}{3}\binom{3}{2}\binom{1}{1} = \frac{12!}{\cancel{10!}\,2!}\,\frac{\cancel{10!}}{\cancel{6!}\,4!}\,\frac{\cancel{6!}}{\cancel{3!}\,3!}\,\frac{\cancel{3!}}{2!\,1!}$

$= \boxed{\dfrac{12!}{2!\,4!\,3!\,2!}}$

b) _ 1 _ 1 _ 5 _ 5 _ 5 _ 9 _

↑ blanks for remaining #s

We have 6 remaining #s to be split by 6 dividers

$= \binom{12}{6}$ ways to distribute remaining #s into spots

Now, we must choose 4 of those #s to be 4:

$= \binom{6}{4}$

$\Rightarrow$ We have $\binom{12}{6}\binom{6}{4}$ such numbers

$= \frac{12!}{6!\,\cancel{6!}} \cdot \frac{\cancel{6!}}{4!\,2!} = \boxed{\dfrac{12!}{6!\,4!\,2!}}$

[redacted]

4. Let S_n denote the number of n-bit strings, with $n \geq 1$, that do not contain the pattern 11.
   (a) (1 point) Compute S_1 and S_2.
   (b) (3 points) Show that S_n satisfies the recurrence relation
   $$S_n = S_{n-1} + S_{n-2}, \qquad \text{for } n \geq 3.$$
   (c) (3 points) Solve the recurrence relation from part (b), i.e. find an explicit formula for S_n in terms of n. You may quote and use any theorems from class. (You do not need to prove by induction that the formula holds if the theorem from class guarantees it.)

a) S_1: 0 or 1, neither contains 11: (2)

S_2: 00, 01, 10, ~~11~~ → (3) do not contain 11

b) Consider an $(n-1)$-bit string. We can construct an n-bit string.

Case 1: We prepend a 0. There are S_{n-1} such strings.

Case 2: We prepend a 1. The second bit must be 0, or else we'd have a string with 11.
∴, there are S_{n-2} such strings.

Adding the two cases, we have $S_n = S_{n-1} + S_{n-2}$.

c) Solve. Second-order linear homogenous w/ constant coefficients.

⇒ $T_n = t^n$ is a solution to S_n.

$$\frac{t^n = t^{n-1} + t^{n-2}}{t^{n-2}} \rightarrow t^2 = t + 1 \rightarrow \boxed{t^2 - t - 1 = 0}$$

$$t = \frac{1 \pm \sqrt{1+4}}{2} \rightarrow \boxed{t = \frac{1+\sqrt{5}}{2},\ t = \frac{1-\sqrt{5}}{2}}$$

→ $A_n = \left(\frac{1+\sqrt{5}}{2}\right)^n$, $B_n = \left(\frac{1-\sqrt{5}}{2}\right)^n$

A linear comb. of A_n and B_n is a solution to S_n.

$S_n = b\left(\frac{1+\sqrt{5}}{2}\right)^n + d\left(\frac{1-\sqrt{5}}{2}\right)^n$

Using initial conditions:

$S_1 = 2 = b\left(\frac{1+\sqrt{5}}{2}\right) + d\left(\frac{1-\sqrt{5}}{2}\right)$ → continue

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$$S_2 = 3 = b\left(\frac{1+\sqrt{5}}{2}\right)^2 + d\left(\frac{1-\sqrt{5}}{2}\right)^2$$

$$b = \frac{3 - d\left(\frac{1-\sqrt{5}}{2}\right)^2}{\left(\frac{1+\sqrt{5}}{2}\right)^2}$$

$$2 = \frac{3 - d\left(\frac{1-\sqrt{5}}{2}\right)^2}{\left(\frac{1+\sqrt{5}}{2}\right)^{2}} \cancel{\left(\frac{1+\sqrt{5}}{2}\right)} + d\left(\frac{1-\sqrt{5}}{2}\right)$$

$$2 = \frac{3}{\left(\frac{1+\sqrt{5}}{2}\right)} - d\left(\frac{\left(\frac{1-\sqrt{5}}{2}\right)^2}{\left(\frac{1+\sqrt{5}}{2}\right)} + \left(\frac{1-\sqrt{5}}{2}\right)\right)$$

$$d = \frac{\frac{3}{\left(\frac{1+\sqrt{5}}{2}\right)} - 2}{\left(\frac{\left(\frac{1-\sqrt{5}}{2}\right)^2}{\left(\frac{1+\sqrt{5}}{2}\right)} + \left(\frac{1-\sqrt{5}}{2}\right)\right)} \rightarrow S_n = b\left(\frac{1+\sqrt{5}}{2}\right)^n + d\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$b = \frac{3 - d\left(\frac{1-\sqrt{5}}{2}\right)^2}{\left(\frac{1+\sqrt{5}}{2}\right)^2}$$

If we use $S_0 = 1$, then we have $1 = b + d$

and $2 = b\left(\frac{1+\sqrt{5}}{2}\right) + d\left(\frac{1-\sqrt{5}}{2}\right)$

$\rightarrow b = 1 - d$

$$2 = (1-d)\left(\frac{1+\sqrt{5}}{2}\right) + d\left(\frac{1-\sqrt{5}}{2}\right)$$

$\downarrow$

easier to solve