21F-MATH61-2 Midterm 2

[redacted]

TOTAL POINTS

20 / 25

QUESTION 1

1 Question 1 5 / 5

✓ **+ 2 pts** **Part a**

✓ **+ 3 pts** **Part b**

+ 0 pts Incorrect

QUESTION 2

2 Question 2 3.5 / 7

+ 7 pts Correct

- 1 pts Left any of your answers in terms of P(n,k) or C(n,k), which the instructions at the beginning of the exam tell you not to

✓ **+ 2 pts** **a) Correct**

+ 1 pts a) Wrote P(26,7)=26!/7!

+ 0.5 pts a) C(26,7) instead of P(26,7)

+ 2 pts b) Correct

- 1 pts b) No work shown

+ 1.5 pts b) Minor error

+ 1 pts b) Partial progress

+ 1 pts b) Only counted those starting (or ending) with "BC" or "CB"

✓ **+ 0.5 pts** **b) A flawed strategy which, if tweaked, might work**

+ 0 pts b) Incorrect

+ 3 pts c) Correct

- 0.5 pts c) Minor error

+ 2 pts c) One of the terms in the inclusion/exclusion is incorrect

- 1 pts c) Did not do inclusion/exclusion correctly, subtracted the "ABC" and "EFG" term instead on adding it

+ 1.5 pts c) Did not do inclusion/exclusion correctly, dropped one of the terms.

✓ **+ 1 pts** **c) Correctly used inclusion/exclusion and gave some attempt at computing the terms but did not do so correctly.**

- 1 pts c) No work shown

+ 0.5 pts c) Correctly used inclusion/exclusion but did not compute any of the terms

+ 0 pts c) Incorrect

1 No, there are $$P(25,6)$$ words if you can only select from the list "A,BC,D,E,...", but that doesn't guarantee BC will be in the word

2 These are not correct

QUESTION 3

3 Question 3 5 / 6

✓ **- 0 pts** **(a) Correct.**

✓ **- 1 pts** **(b) Mostly correct, with some mistakes.**

3 7-1, twice

QUESTION 4

4 Question 4 6.5 / 7

✓ **+ 1 pts** **(a): Correct**

✓ **+ 3 pts** **(b): Correct**

+ 2 pts (b): Partial solution

+ 0 pts (b): Totally incorrect/no meaningful progress

+ 3 pts (c): Correct

+ 3 pts (c): Correct method, error in calculation

✓ **+ 2.5 pts** **(c): Correct method, easy-to-spot error in solution**

+ 2.5 pts (c): Correct, but no explicit formula

+ 2 pts (c): Partial solution

+ 1.5 pts (c): Partially correct method

+ 0 pts (c): Incorrect/no meaningful progress

+ 0 pts (a): Incorrect

+ 1 pts (b): Some progress

+ 1 pts (c): Some progress

4 Sanity check: this is false for $$n=0,2$$.

Midterm 2

Name: [redacted]

UID:

Instructions: Do not open this exam until instructed to do so. You will have 50 minutes to complete the exam. Please print your name and student ID number above. You may **not** use calculators, books, notes, or any other material to help you. Please make sure your phone is silenced and stowed where you cannot see it. You may use the front and back of the page for your answers. **Do not write answers for one question on the page of another question**. If you need scratch paper, please ask one of the proctors. You must **show all your work** to receive credit.

Note: In this entire exam, you may leave your responses written as products, fractions, and with factorials. However, you are not allowed to leave expressions of the form $C(n, k)$, $P(n, k)$, etc. As always, you need to justify all your answers.

Question	Points	Score
1	5	
2	7	
3	6	
4	7	
Total:	25	

UID: [redacted]

1. Suppose Matt the baker sells 5 types of bread, 7 types of pastries and 3 types of tarts. All his clients buy exactly one bread, one pastry and one tart every time they visit the bakery.

   (a) (2 points) If in one day, 150 people show up at the bakery, show that there are at least 2 people who bought the exact same bread, pastry and tart.

   (b) (3 points) How many people would have to show up to make sure that at least 5 people buy the same bread and pastry?

a. There are $\underset{\text{bread}}{5} \cdot \underset{\text{pastries}}{7} \cdot \underset{\text{tarts}}{3} = 115$ different combinations of 1 bread, pastry, and tart. ⇒ 115

By the Pigeonhole Principle, there is at least 2 people who have the same exact bread, pastry, and tart since 150 people > 115 combinations

b. $5 \cdot 7 = 35$ combinations of bread and pastry

Assume every combination has 4 people

$35 \cdot 4 = 140$ people

Then by Pigeonhole principle if there are 141 people, at least 5 will buy the same bread and pastry.

$\left\lceil \frac{141}{140} \right\rceil = 5$

UID: [redacted]

2. (a) (2 points) How many words of 7 letters can we form with the 26 letters of the Latin alphabet if we don't allow letters to be repeated in a word?

   (b) (2 points) How many such words can we form if we require B and C to appear in it next to each other (in either order)?

   (c) (3 points) How many such words (as in part (a)) can we form if we don't allow the strings ABC or EFG to appear in it?

a. $26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20$
1st 2nd 3rd 4th 5th 6th 7th

b. let X = a string of B and C in any order
⇒ 2! ways to arrange B and C
A desired word can be form with the remaining 24 letters and X, of a total length of 7
25 values
$2! \cdot (25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20)$

c. let x = "ABC",
⇒ $24 \cdot 23 \cdot 22 \cdot 21 \cdot 20$
possible combinations with ABC
let y = EFG
⇒ $24 \cdot 23 \cdot 22 \cdot 21 \cdot 20$
possible combination with EFG
with both x and y
⇒ $22 \cdot 21 \cdot 20$

⇒ by inclusion - exclusion, strings without ABC or EFG
$26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 - [24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 + 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 - 22 \cdot 21 \cdot 20]$

1

2

UID: [redacted]

3. (a) (3 points) How many distinct 12-digit numbers can we form with the digits 1, 1, 4, 4, 4, 4, 5, 5, 5, 8, 8, 9?

   (b) (3 points) In how many of those numbers do the odd digits appear in increasing order? (For instance, 144185584594 is such a number.)

a, 2 1's
4 4's
3 5's
2 8's
1 9

$\Rightarrow C(12,2)\cdot C(10,4)\cdot C(6,3)\cdot C(3,2)\cdot C(1,1)$

$= \dfrac{12!}{2!\cdot 4!\cdot 3!\cdot 2!\cdot 1!}$

b. $\dfrac{6!}{4!2!}$ ways to order 4 4s and 2 8s

Drop each 4 or 8 into buckets in between the odd numbers in sequence (7 buckets) $\Rightarrow C(6+7, 7)$

$\dfrac{13!}{6!\cdot 7!}\cdot\dfrac{6!}{4!2!}$

4|444|8|8|

. 1 . 1 . 5 . 5 5 . 9

$\frac{1+\sqrt{5}}{2} \cdot \frac{1+\sqrt{5}}{2} = \frac{1+2\sqrt{5}+5}{4} = \frac{6+2\sqrt{5}}{4} = \frac{3+\sqrt{5}}{2}$

UID: [redacted]

4. Let S_n denote the number of n-bit strings, with $n \geq 1$, that do not contain the pattern 11.
   (a) (1 point) Compute S_1 and S_2.
   (b) (3 points) Show that S_n satisfies the recurrence relation

$$S_n = S_{n-1} + S_{n-2}, \qquad \text{for } n \geq 3.$$

   (c) (3 points) Solve the recurrence relation from part (b), i.e. find an explicit formula for S_n in terms of n. You may quote and use any theorems from class. (You do not need to prove by induction that the formula holds if the theorem from class guarantees it.)

a. $S_1 = 2$, $S_2 = 3$

b. A string that does not contain the pattern 11 must either
1) start with 0 $\Rightarrow$ 0 _ _ _ _ ... _ (n-1 string without 11)
2) start with 10 $\Rightarrow$ 10 _ _ _ _ ... _ (n-2 string without 11)

Then

$S_n = S_{n-1} + S_{n-2}$ if $n \geq 3$

c. If a recurrence relation is homogeneous and of order 2 then $V_n = t^n$ is a solution and any linear combination of t_1^n and t_2^n is also a solution if both t_1^n and t_2^n are solutions

$\Rightarrow t^n = t^{n-1} + t^{n-2}$

$\Rightarrow t^2 = t + 1$

$\Rightarrow t^2 - t - 1 = 0$

$\Rightarrow t = \frac{1 \pm \sqrt{5}}{2}$

$\Rightarrow S_n = b\left(\frac{1+\sqrt{5}}{2}\right)^n + d\left(\frac{1-\sqrt{5}}{2}\right)^n$

$\Rightarrow S_1 = 2 = b\left(\frac{1+\sqrt{5}}{2}\right) + d\left(\frac{1-\sqrt{5}}{2}\right)$

$S_2 = 3 = b\left(\frac{1+\sqrt{5}}{2}\right)^2 + d\left(\frac{1-\sqrt{5}}{2}\right)^2$

continue on back

[redacted]

$$\left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{1+2\sqrt{5}+5}{4} = \frac{3+\sqrt{5}}{2}$$

$$\left(\frac{1-\sqrt{5}}{2}\right)^2 = \frac{1-2\sqrt{5}+5}{4} = \frac{3-\sqrt{5}}{2}$$

$$b\left(\frac{1+\sqrt{5}}{2}\right) + d\left(\frac{1-\sqrt{5}}{2}\right) = 2$$

$$b\left(\frac{3+\sqrt{5}}{2}\right) + d\left(\frac{3-\sqrt{5}}{2}\right) = 3$$

$$\Rightarrow \cancel{b\left(\frac{1-5}{4}\right)} - b\left(\frac{3+\sqrt{5}}{2}\right) = 2\left(\frac{1-\sqrt{5}}{2}\right) - 3$$

$$b\left(1 - \frac{3+\sqrt{5}}{2}\right) = 2\left(\frac{1-\sqrt{5}}{2}\right) - 3$$

$$b = \frac{-2-\sqrt{5}}{1-\frac{3+\sqrt{5}}{2}} \quad \Rightarrow b = \frac{-4-2\sqrt{5}}{-1-\sqrt{5}} = 2$$

$$\Rightarrow 1+\sqrt{5} + d\left(\frac{1-\sqrt{5}}{2}\right) = 2$$

$$d\left(\frac{1-\sqrt{5}}{2}\right) = 1-\sqrt{5}$$

$$d = 2$$

$$\Rightarrow \boxed{S_n = 2\left(\frac{1\pm\sqrt{5}}{2}\right)^n + 2\left(\frac{1\pm\sqrt{5}}{2}\right)^n}$$