21F-MATH61-2 Midterm 2

QUAN DO

TOTAL POINTS

25 / 25

QUESTION 1

1 Question 1 5 / 5

✓ + **2 pts** Part a

✓ + **3 pts** Part b

+ **0 pts** Incorrect

QUESTION 2

2 Question 2 7 / 7

✓ + **7 pts** Correct

- **1 pts** Left any of your answers in terms of P(n,k) or C(n,k), which the instructions at the beginning of the exam tell you not to

+ **2 pts** a) Correct

+ **1 pts** a) Wrote P(26,7)=26!/7!

+ **0.5 pts** a) C(26,7) instead of P(26,7)

+ **2 pts** b) Correct

- **1 pts** b) No work shown

+ **1.5 pts** b) Minor error

+ **1 pts** b) Partial progress

+ **1 pts** b) Only counted those starting (or ending) with "BC" or "CB"

+ **0.5 pts** b) A flawed strategy which, if tweaked, might work

+ **0 pts** b) Incorrect

+ **3 pts** c) Correct

- **0.5 pts** c) Minor error

+ **2 pts** c) One of the terms in the inclusion/exclusion is incorrect

- **1 pts** c) Did not do inclusion/exclusion correctly, subtracted the "ABC" and "EFG" term instead on adding it

+ **1.5 pts** c) Did not do inclusion/exclusion correctly, dropped one of the terms.

+ **1 pts** c) Correctly used inclusion/exclusion and gave some attempt at computing the terms but did not do so correctly.

- **1 pts** c) No work shown

+ **0.5 pts** c) Correctly used inclusion/exclusion but did not compute any of the terms

+ **0 pts** c) Incorrect

QUESTION 3

3 Question 3 6 / 6

✓ - **0 pts** (a) Correct.

✓ - **0 pts** (b) Correct.

QUESTION 4

4 Question 4 7 / 7

✓ + **1 pts** (a): Correct

✓ + **3 pts** (b): Correct

+ **2 pts** (b): Partial solution

+ **0 pts** (b): Totally incorrect/no meaningful progress

✓ + **3 pts** (c): Correct

+ **3 pts** (c): Correct method, error in calculation

+ **2.5 pts** (c): Correct method, easy-to-spot error in solution

+ **2.5 pts** (c): Correct, but no explicit formula

+ **2 pts** (c): Partial solution

+ **1.5 pts** (c): Partially correct method

+ **0 pts** (c): Incorrect/no meaningful progress

+ **0 pts** (a): Incorrect

+ **1 pts** (b): Some progress

+ **1 pts** (c): Some progress

Midterm 2

Name: Quan Do

UID: [redacted]

Instructions: Do not open this exam until instructed to do so. You will have 50 minutes to complete the exam. Please print your name and student ID number above. You may **not** use calculators, books, notes, or any other material to help you. Please make sure your phone is silenced and stowed where you cannot see it. You may use the front and back of the page for your answers. **Do not write answers for one question on the page of another question.** If you need scratch paper, please ask one of the proctors. You must **show all your work** to receive credit.

Note: In this entire exam, you may leave your responses written as products, fractions, and with factorials. However, you are not allowed to leave expressions of the form $C(n, k)$, $P(n, k)$, etc. As always, you need to justify all your answers.

Question	Points	Score
1	5	
2	7	
3	6	
4	7	
Total:	25	

UID: [redacted]

1. Suppose Matt the baker sells 5 types of bread, 7 types of pastries and 3 types of tarts. All his clients buy exactly one bread, one pastry and one tart every time they visit the bakery.

   (a) (2 points) If in one day, 150 people show up at the bakery, show that there are at least 2 people who bought the exact same bread, pastry and tart.

   (b) (3 points) How many people would have to show up to make sure that at least 5 people buy the same bread and pastry?

a) There are $5 \cdot 7 \cdot 3 = 105$ combinations of bread / pastry / tart. Each person gets one combination. 150 people > 105 combinations, so by the pigeonhole principle, there are at least 2 people with the same combination.

b) # of combinations : $5 \cdot 7 = 35$

$5 \leq \lceil \frac{n}{35} \rceil$

$n \geq 4 \cdot 35 + 1 \rightarrow$ $n \geq 141$ people

UID: [redacted]

2. (a) (2 points) How many words of 7 letters can we form with the 26 letters of the Latin alphabet if we don't allow letters to be repeated in a word?

(b) (2 points) How many such words can we form if we require B and C to appear in it next to each other (in either order)?

(c) (3 points) How many such words (as in part (a)) can we form if we don't allow the strings ABC or EFG to appear in it?

a) Choose 7 from 26: $\binom{26}{7}$

Permute them: 7! $\longrightarrow \binom{26}{7}7! = \boxed{\frac{26!}{19!}}$

b) Consider BC to be one letter. Now there are 24 letters, and we need to choose 5: $\binom{24}{5}$. Now permute them: 6!

Now order BC (2 possibilities)

$\rightarrow$ Total: $\binom{24}{5} \cdot 6! \cdot 2 = \frac{24!}{19!\,5!} 6! \cdot 2 = \boxed{\frac{24!}{19!} \cdot 12}$

c) # of words with ABC: _ _ ABC _ _ $\rightarrow 5 \cdot P(23,4)$
1 2 3 4 5

# of words with EFG: Also $5 \cdot P(23,4)$

# of words with both ABC and EFG: ABC _ EFG
ABCEFG_ $\cdot 2 \cdot 20 = 120$
_ABCEFG

↑ swap them ↑ last letter

Total disallowed words $= 10 \cdot P(23,4) - 120$

Total allowed $= P(26,7) - 10 \cdot P(23,4) + 120$

$= \boxed{\frac{26!}{19!} - 10 \cdot \frac{23!}{19!} + 120}$

UID: [redacted]

3. (a) (3 points) How many distinct 12-digit numbers can we form with the digits 1, 1, 4, 4, 4, 4, 5, 5, 5, 8, 8, 9?

   (b) (3 points) In how many of those numbers do the odd digits appear in increasing order? (For instance, 144185584594 is such a number.)

a) $\boxed{\dfrac{12!}{2!\,4!\,3!\,2!}}$ We are ordering $n_1=2, n_2=4, n_3=3, n_4=2, n_5=1$ identical elements, so there are $\dfrac{(n_1+n_2+\dots+n_5)!}{n_1!\,n_2!\,n_3!\,n_4!\,n_5!}$ orderings.

b) 6 odd digits: 1, 1, 5, 5, 5, 9

Total orderings of odd digits: $\dfrac{6!}{2!\,3!} = \dfrac{6\cdot5\cdot4}{2} = 60$

# of orderings with increasing digits: 1

$\rightarrow \boxed{\dfrac{1}{60}\cdot\dfrac{12!}{2!\,4!\,3!\,2!}}$

4. Let S_n denote the number of n-bit strings, with $n \geq 1$, that do not contain the pattern 11.

   (a) (1 point) Compute S_1 and S_2.

   (b) (3 points) Show that S_n satisfies the recurrence relation

   $$S_n = S_{n-1} + S_{n-2}, \qquad \text{for } n \geq 3.$$

   (c) (3 points) Solve the recurrence relation from part (b), i.e. find an explicit formula for S_n in terms of n. You may quote and use any theorems from class. (You do not need to prove by induction that the formula holds if the theorem from class guarantees it.)

a) $S_1 = 2$, $S_2 = 3$

b) An n-bit string can start with 0 or 1.

If 0: 0 _ _ _ ... (n-1 bits)

→ The only condition for these n-1 bits is that they don't contain 11. So there are S_{n-1} possibilities.

If 1: 10 _ _ _ ... (n-2 bits) → S_{n-2} possibilities for the same reason.

(has to be zero)

Total: $S_n = S_{n-1} + S_{n-2}$

c) $t^2 = c_1 t + c_2 = t + 1$

$t^2 - t - 1 = 0 \rightarrow t = \frac{1 \pm \sqrt{5}}{2}$ $bt^n + dt^n$ is a solution

$b\left(\frac{1+\sqrt{5}}{2}\right)^1 + d\left(\frac{1-\sqrt{5}}{2}\right)^1 = 2$

$b\left(\frac{1+\sqrt{5}}{2}\right)^2 + d\left(\frac{1-\sqrt{5}}{2}\right)^2 = 3$

$b = \dfrac{2 - d\left(\frac{1-\sqrt{5}}{2}\right)}{\frac{1+\sqrt{5}}{2}}$

$\left(2 - d\left(\frac{1-\sqrt{5}}{2}\right)\right)\left(\frac{1+\sqrt{5}}{2}\right) + d\left(\frac{1-\sqrt{5}}{2}\right)^2 = 3$

$1+\sqrt{5} - d\left(\frac{1-\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right) + d\left(\frac{1-\sqrt{5}}{2}\right)^2 = 3$

$1+\sqrt{5} - d\left(\frac{1-\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right) = 3$

$d = -\dfrac{3-(1+\sqrt{5})}{\left(\frac{1-\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right)}$ $b = \dfrac{2 + \dfrac{3-(1+\sqrt{5})}{\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right)}}{\frac{1+\sqrt{5}}{2}}$

$S_n = \left(2 + \dfrac{3-(1+\sqrt{5})}{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\dfrac{3-(1+\sqrt{5})}{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}$