22S-MATH-61-LEC-1 Midterm 1

TOTAL POINTS

20 / 20

QUESTION 1

1 Question 1 5 / 5

✓ **- 0 pts** Correct

QUESTION 2

2 Question 2 4 / 4

✓ **- 0 pts** Correct

QUESTION 3

3 Question 3 6 / 6

✓ **- 0 pts** (a) Correct.

✓ **- 0 pts** (b) Correct.

✓ **- 0 pts** (c) Correct.

✓ **- 0 pts** (d) Correct.

1 The existence is the only important thing here

QUESTION 4

4 Question 4 5 / 5

✓ **+ 2 pts** $$f$$ is surjective

✓ **+ 1 pts** $$f$$ is not injective

✓ **+ 1 pts** $$g$$ that works

✓ **+ 1 pts** justification that $$g$$ works

+ 0 pts No substantial progress

1. (5 points) Prove by induction that

$$1! \cdot 1 + 2! \cdot 2 + \ldots + n! \cdot n = (n+1)! - 1$$

for every integer $n \geq 1$.

Proof:

Basis step: $n=1$, LHS $= 1! \times 1 = 1$, RHS $= (1+1)! - 1$
$= 2! - 1 = 1$ ✓

Inductive step: Assume for every integer $n \geq 1$, $1! \cdot 1 + 2! \cdot 2 + \ldots + n! \cdot n = (n+1)! - 1$

for $n+1$:

$1! \cdot 1 + 2! \cdot 2 + \cdots + n! \cdot n + (n+1)! \cdot (n+1)$

$= (n+1)! - 1 + (n+1)! \cdot (n+1)$

$= (n+1)! \cdot (n+1+1) - 1$

$= (n+2)! - 1$

$= ((n+1)+1)! - 1$

the inductive step is complete.

Therefore, for every integer $n \geq 1$, $1! \cdot 1 + 2! \cdot 2 + \cdots + n! \cdot n = (n+1)! - 1$

$\square$

2. (4 points) Prove the following statement:

For every integer $n \geq 1$: If $3^n = n^2 + 2n$, then n is odd.

Proof (by contrapositive): 　　　　　　　　　　　　If P, then Q
$\Rightarrow$ if not Q, then not P.

Assume for contrapositive that n is even and $n \geq 1$

i.e. $\exists\, k \in \mathbb{Z}^+ : n = 2k$

$LHS: 3^n = 3^{2k} = 9^k$

$RHS = n^2 + 2n = (2k)^2 + 2(2k) = 4k^2 + 4k = 2(2k^2 + 2k)$

LHS is an odd number, because an odd number raised to any power would still be an odd number.

RHS is an even number. Since $k, 2k \in \mathbb{Z}^+$, then $\exists\, p \in \mathbb{Z}^+ : 2k^2 + 2k = p$, then $2(2k^2 + 2k) = 2p$, which is an even integer.

Since an even integer and an odd integer cannot be equal to each other, $LHS \neq RHS$.

Therefore, if n is not odd, then $3^n \neq n^2 + 2n$.

Therefore, by contrapositive, if $3^n = n^2 + 2n$ for every integer $n \geq 1$, then n is odd.

$\square$

3. Let $f : X \to Y$ be a function.

(a) (1 point) Define what it means for f to be one-to-one (injective).

(b) (1 point) Define what it means for f to be onto (surjective).

Let now $f : X \to Y$ and $g : Y \to Z$ be functions such that $g \circ f$ is bijective.

$g(f(x))$

(c) (2 points) Show that f is one-to-one.

(d) (2 points) Show that g is onto.

a) f is injective if $\forall x_1, x_2 \in X$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

b) f is surjective if $\forall y \in Y$, $\exists x \in X$: $f(x) = y$.

c) $g \circ f$ is bijective $\Rightarrow$ $g \circ f$ is both one-to-one and onto.

f is one-to-one

proof:

Let ~~there~~ be $x_1, x_2 \in X$: $x_1 \neq x_2$

Since $g \circ f$ is one-to-one, $g \circ f(x_1) \neq g \circ f(x_2) \Rightarrow g(f(x_1)) \neq g(f(x_2))$

Since g is a function, $f(x_1) \neq f(x_2)$.

Therefore, f is one-to-one.

$\square$

d) g is onto ($\forall z \in Z$: $\exists y \in Y$: $g(y) = z$)

proof:

Since $g \circ f$ is onto: $\forall z \in Z$, $\exists x \in X$: $g \circ f(x) = z \Rightarrow g(f(x)) = z$.

Since f is a function: $\forall x \in X$, there exists one and only one $y \in Y$ such that $f(x) = y$.

$\Rightarrow \forall z \in Z$, $\exists y \in Y$: $g(y) = z$

Therefore, g is onto.

$\square$

4. Let $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be defined by $f(m, n) = n + 5$.

integer

(a) (3 points) Show that f is onto, but not one-to-one.

(b) (2 points) Find a function $g : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ such that $f \circ g$ is bijective. Justify your answer.

Let $f: X \to Y$. Define domain $\mathbb{Z} \times \mathbb{Z}$ to be X and codomain $\mathbb{Z}$ to be Y

a) f is not one-to-one:

$\exists x_1, x_2 \in X : f(x_1) = f(x_2)$ but $x_1 \neq x_2$.

Example: $x_1 = (1,2)$, $f(1,2) = 2+5 = 7$

$x_2 = (2,2)$, $f(2,2) = 2+5 = 7$

$(1,2) \neq (2,2) \Rightarrow f$ is not one-to-one.

f is onto:

Proof: $f(m,n) = y = n+5$

$n = y-5$.

Since $n \in \mathbb{Z}$ and $y \in \mathbb{Z}$, $\forall y \in Y\ \exists n \in \mathbb{Z} : f(m,n) = y$.

In particular, the value of m does not matter as long as $m \in \mathbb{Z}$, and $n = y-5$. therefore, f is onto. □

b) $f \circ g$ being bijective: $f \circ g$ is both one-to-one and onto.

g: $g(K) = (K, K+1)$, let its domain be K.

$f \circ g : f(g(K)) = f(K, K+1) = K+6$.

Proof that $f \circ g$ is bijective:

$f \circ g$ is onto: $\forall y \in Y, \exists k \in K : f \circ g(k) = y$. In particular, $k = y - 6$

$f \circ g$ is one-to-one: let there be $k_1, k_2 \in K : f \circ g(k_1) = f \circ g(k_2)$

$f \circ g(k_1) = f \circ g(k_2)$

$k_1 + 6 = k_2 + 2$

$k_1 = k_2$

Therefore, $f \circ g$ is bijective. □