21F-MATH61-2 Midterm 1

[redacted]

TOTAL POINTS

19 / 25

QUESTION 1

1 Question 1 **3.5 / 6**

✓ **- 1 pts** **Need to specify what $$n$$ (or $$k$$) is in the inductive step, namely an integer $$\mathbf{\geq 3}$$.**

✓ **- 1.5 pts** **A proof should not start with the thing you have to prove. If you do, it has to be clearly explained why all lines/equations are equivalent.**

QUESTION 2

2 Question 2 **4 / 7**

+ **7 pts** Correct

✓ **+ 2 pts** **a) Correct**

+ **2 pts** a) Did not construct a counterexample, but showed they would know what such a counterexample would be

+ **1.5 pts** a) Did not construct a counterexample, but ALMOST showed they would know what such a counterexample would be

+ **1 pts** a) Claimed $$f$$ is not one-to-one because the attempt to prove it does not work.

+ **1 pts** a) Poor proof, but vaguely correct explanation

+ **0 pts** a) Incorrect

+ **2 pts** b) Correct

✓ **+ 0 pts** **b) Incorrect**

✓ **+ 2 pts** **c) Correct**

+ **1.5 pts** c) minor error, or mostly correct explanation that isn't quite a proof

+ **1 pts** c) Poor proof, but vaguely correct explanation

+ **0.5 pts** c) Some idea of what to do, but does not know how to properly explain their thinking.

+ **0 pts** c) Incorrect

+ **1 pts** d) Correct

+ **0.5 pts** d) $$g\circ f$$ is onto, but $$g$$ is not a function from $$\mathbb{Z}$$ to $$\mathbb{Z}$$. For example, $$g(x)=\frac{1}{2}x$$ has $$g(1)=\frac{1}{2}\notin \mathbb{Z}$$

✓ **+ 0 pts** **d) Incorrect**

1 $$m\notin\mathbb{Z}$$, for example, if $$y=1$$

2 $$f$$ is not onto

QUESTION 3

3 Question 3 **7 / 7**

✓ **+ 2 pts** **Part a**

✓ **+ 2 pts** **Part b**

✓ **+ 3 pts** **Part c**

+ **0 pts** Incorrect

QUESTION 4

4 Question 4 **4.5 / 5**

✓ **- 0.5 pts** **(b): Correct approach with error**

3 What about $$p=n$$?

Midterm 1

Name: [redacted]

UID: [redacted]

Instructions: Do not open this exam until instructed to do so. You will have 50 minutes to complete the exam. Please print your name and student ID number above. You may **not** use calculators, books, notes, or any other material to help you. Please make sure your phone is silenced and stowed where you cannot see it. You may use the front and back of the page for your answers. **Do not write answers for one question on the page of another question**. If you need scratch paper, please ask one of the proctors. You must **show all your work** to receive credit.

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1. Prove by induction that

$$2n + 1 < 2^n$$

for every $n \geq 3$.

Proof:

Base Case: $n=3$: $2(3)+1=7$

$2^3 = 8$

$7 < 8$ ✓

Inductive Step:

Assume that $2n+1 < 2^n$ is true.

Consider $n+1$:

$2(n+1)+1 = (2n+1)+2$

$2^{n+1} = 2^n \cdot 2 = 2^n + 2^n$

$\rightarrow (2n+1)+2 \overset{?}{<} 2^n + 2^n$

$\rightarrow 2n+1 \overset{?}{<} 2^n + (2^n - 2)$

Because $n \geq 3$, $2^n > 2$

$(2^3 = 8, 8 > 2)$.

By our inductive hypothesis, $2n+1 < 2^n$ holds, so if we add a positive quantity $2^n - 2$ to the right side, the right side will still be greater than the left side, and by induction, $2n+1 < 2^n$ for all $n \geq 3$.

[redacted]

2. Let $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be defined by $f(m,n) = 2m$.
   (a) Is f one-to-one?
   (b) Is f onto?
   (c) Does there exist a function $g : \mathbb{Z} \to \mathbb{Z}$ such that $g \circ f$ is one-to-one?
   (d) Does there exist a function $g : \mathbb{Z} \to \mathbb{Z}$ such that $g \circ f$ is onto?

   Justify your answers.

a) No. For example, $f(3,5) = 6$, and $f(3,1) = 6$, but $(3,5) \neq (3,1)$.

One to one means $\forall x_1, x_2 \in X,\ f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.
In this case, $x_1 \neq x_2$.

b) Yes. We can write $f(m,n) = 2m = y$.
Then, $m = \frac{y}{2}$, and $n \in \mathbb{Z}$.
$\therefore \exists (m,n) \in \mathbb{Z} \times \mathbb{Z} : f(m,n) = y$. $(m,n) = (\frac{y}{2}, n)$ for all $y, n \in \mathbb{Z}$.

c) No. Seeing that f is not one to one,
$\exists (m_1,n_1), (m_2,n_2) \in \mathbb{Z} \times \mathbb{Z} : f(m_1,n_1) = f(m_2,n_2)$ where $(m_1,n_1) \neq (m_2,n_2)$.
Then, $g(f(m_1,n_1)) = g(f(m_2,n_2))$, and $(m_1,n_1) \neq (m_2,n_2)$.
$\therefore \exists (m_1,n_1), (m_2,n_2) \in \mathbb{Z} \times \mathbb{Z}$ such that $g(f(m_1,n_1)) = g(f(m_2,n_2))$ where $(m_1,n_1) \neq (m_2,n_2)$.
$\therefore g \circ f$ is not one to one.

d) Yes. Because f is onto, Range(f) $= \mathbb{Z}$.
Therefore, any function g that is onto will result in gof being onto as the composition of onto functions is onto (proven in HW).

UID: [redacted]

3. (a) Let X be a nonempty set, and $Y \subseteq X$ be a subset of X. Consider the relation R on $\mathcal{P}(X)$ defined by $A\,R\,B$ if and only if $A \cap Y = B \cap Y$. Show that R is an equivalence relation.

   (b) Suppose $X = \{1, 2, 3, 4, 5, 6\}$ and $Y = \{1, 3, 4, 5\}$. Compute the equivalence class of $A = \{1, 2, 3\}$ for the equivalence relation defined in part (a).

   (c) Suppose S is an equivalence relation on a nonempty set Z such that S is also a partial order. Show that $S = \{(x, x) \mid x \in Z\}$ (in other words, every element is related to itself but to no other elements).

a) ARB iff $A\cap Y = B \cap Y$.

- Reflexive: $A \cap Y = A \cap Y$ for all A. Therefore, ARA holds for all A. The relation is reflexive.
- Symmetric: $A \cap Y = B \cap Y$ implies $B \cap Y = A \cap Y$ for all A, B by definition of $=$. $\therefore ARB \Rightarrow BRA$. The relation is symmetric.
- Transitive: If $A \cap Y = B \cap Y$ and $B \cap Y = C \cap Y$, by transitivity of set equivalence, $A \cap Y = C \cap Y$.
  $\therefore$ ARB, and $BRC \Rightarrow ARC$, and the relation is transitive.

Because R is reflexive, symmetric, and transitive, it is an equivalence relation.

b) $X = \{1, 2, 3, 4, 5, 6\}$ $Y = \{1, 3, 4, 5\}$

$A = \{1, 2, 3\} \rightarrow A \cap Y = \{1, 3\}$

$|A| = \{B \mid B \subseteq X \text{ and } B \cap Y = \{1, 3\}\}$

$|A| = \{\{1, 2, 3\}, \{1, 3\}, \{1, 3, 6\}, \{1, 2, 3, 6\}\}$

→

c) Suppose S is an equivalence relation and a partial order, and that there exists some $x, y \in \mathbb{Z}$ such that xRy where $x \neq y$.

Because S is an equivalence relation, it is symmetric. $\therefore xRy \Rightarrow yRx$, and (x, y) and $(y, x) \in S$.

Because S is a partial order, S is anti-symmetric. $\therefore xRy$ and $yRx \Rightarrow \underline{x = y}$.

↳ Contradiction

We assumed that $x \neq y$, but this contradicts that.

$\therefore$ By contradiction, if S is an equivalence relation and a partial order, $S = \{(x, x) \mid x \in \mathbb{Z}\}$

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4. Let X and Y be finite sets, with $|X| = n$, $|Y| = p$.
   (a) How many one-to-one functions are there from X to Y?
   (b) How many reflexive relations are there on X?

   Justify your answers.

a) Pick an arbitrary $x_1 \in X$. Then, there are p options to map to. Because the functions must be one-to-one, each y can only have one x mapped to it. Therefore, for the next $x_2 \in X$, there are $p-1$ possibilities.

In total, there are

$$p(p-1)(p-2)\ldots(p-(n-1))$$

possibilities for $p > n$.

If $p < n$, there are 0 functions as you cannot possibly map each x to a unique y if $p < n$.

ex: $X = \{1,2,3\}$
$Y = \{1,2,3,4,5\}$
num functions $= 5 \cdot 4 \cdot 3 = p \cdot (p-1) \cdot (p-2)$

3

b) Assuming we say R, a relation on X is reflexive, then once we include all reflexive pairs, each of the n $x \in X$ has $n-1$ possible additional relations that it can either (1) include or (2) not include (two options for each $n-1$ other x). There are therefore $n \cdot 2^{n-1}$ reflexive relations on X.

ex: $X = \{1,2\}$

$(1,2)$, $(2,1)$ $= 4 = 2 \cdot 2^1$ ✓
$(1,2)$ and $(2,1)$,
neither