21F-MATH61-2 Midterm 1

TOTAL POINTS

22.5 / 25

QUESTION 1

1 Question 1 6 / 6

✓ **- 0 pts** **Correct**

QUESTION 2

2 Question 2 5.5 / 7

+ **7 pts** Correct

✓ **+ 2 pts** **a) Correct**

+ **2 pts** a) Did not construct a counterexample, but showed they would know what such a counterexample would be

+ **1.5 pts** a) Did not construct a counterexample, but ALMOST showed they would know what such a counterexample would be

+ **1 pts** a) Claimed $$f$$ is not one-to-one because the attempt to prove it does not work.

+ **1 pts** a) Poor proof, but vaguely correct explanation

+ **0 pts** a) Incorrect

✓ **+ 2 pts** **b) Correct**

+ **0 pts** b) Incorrect

✓ **+ 2 pts** **c) Correct**

+ **1.5 pts** c) minor error, or mostly correct explanation that isn't quite a proof

+ **1 pts** c) Poor proof, but vaguely correct explanation

+ **0.5 pts** c) Some idea of what to do, but does not know how to properly explain their thinking.

+ **0 pts** c) Incorrect

+ **1 pts** d) Correct

+ **0.5 pts** d) $$g\circ f$$ is onto, but $$g$$ is not a function from $$\mathbb{Z}$$ to $$\mathbb{Z}$$. For example, $$g(x)=\frac{1}{2}x$$ has $$g(1)=\frac{1}{2}\notin \mathbb{Z}$$

✓ **+ 0 pts** **d) Incorrect**

- 0.5 **Point adjustment**

b) Minor error

1 $$m=\frac{3}{2}$$

2 You should really provide an explicit counterexample.

QUESTION 3

3 Question 3 7 / 7

✓ **+ 2 pts** **Part a**

✓ **+ 2 pts** **Part b**

✓ **+ 3 pts** **Part c**

+ **0 pts** Incorrect

QUESTION 4

4 Question 4 4 / 5

✓ **- 1 pts** **(b): Right general approach, erroneous execution**

3 Which gets you $$2^{n(n-1)}$$ reflexive relations.

Midterm 1

Name: ______________________

UID: ______________________

Instructions: Do not open this exam until instructed to do so. You will have 50 minutes to complete the exam. Please print your name and student ID number above. You may **not** use calculators, books, notes, or any other material to help you. Please make sure your phone is silenced and stowed where you cannot see it. You may use the front and back of the page for your answers. **Do not write answers for one question on the page of another question**. If you need scratch paper, please ask one of the proctors. You must **show all your work** to receive credit.

UID: ________

1. Prove by induction that

$$2n + 1 < 2^n$$

for every $n \geq 3$.

Basis step: Let $n = n_0 = 3$. $2 \cdot 3 + 1 = 7 < 2^3 = 8$ ✓

Inductive step: Let $k \geq 3$. Assume $2k+1 < 2^k$.

We must show $2(k+1) + 1 < 2^{(k+1)}$

By assumption, $2k+1 < 2^k$. We can then reason that $2 \cdot (2k+1) < 2 \cdot 2^k$.

- $2 \cdot (2k+1) < 2 \cdot 2^k$ is equivalent to $4k+2 < 2^{k+1}$

We must now show that $2(k+1)+1 < 4k+2$ for $k \geq 3$.

- $4k+2 = 2(k+1) + 2k$
- Since $k \geq 3$, it must be true that $2k > 1$.
- Add $2(k+1)$ to both sides of the inequality $1 < 2k$.
  - We get $2(k+1) + 1 < 2(k+1) + 2k$

We have proven $2(k+1) + 1 < 4k+2$.

By the transitivity, $2(k+1)+1 < 2^{k+1}$ ✓

UID: ________

integers
cartesian product x integers

2. Let $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be defined by $f(m, n) = 2m$.

   (a) Is f one-to-one? $\forall x_1, x_2$ if $f(x_1) = f(x_2)$, $x_1 = x_2$

   (b) Is f onto? $\forall y \in Y \; \exists x \in X$ s.t. $f(x) = y$

   (c) Does there exist a function $g : \mathbb{Z} \to \mathbb{Z}$ such that $g \circ f$ is one-to-one?

   (d) Does there exist a function $g : \mathbb{Z} \to \mathbb{Z}$ such that $g \circ f$ is onto?

   Justify your answers.

a) No, see counterexample: $(m_1, n_1) = (2,1)$ and $(m_2, n_2) = (2,0)$

$f(m_1, n_1) = 2 \cdot 2 = 4$ $f(m_2, n_2) = 4$

Despite (m_1, n_1) and (m_2, n_2) not being equal, $f(m_1, n_1) = f(m_2, n_2)$

b) No, see counterexample: Let $y = 3$. The co-domain is all + integers, so if f is onto y must have an (m,n) s.t. $f(m,n) = y$.

We have $3 = 2m \Rightarrow m = \frac{2}{3}$. But the domain is $\mathbb{Z} \times \mathbb{Z}$, and $\frac{2}{3} \notin \mathbb{Z}$, so there does not exist an (m,n) in the domain $\mathbb{Z} \times \mathbb{Z}$ s.t. $f(m,n) = 3$

c) $g \circ f = g(f(m,n)) = g(2m)$

For g of to be one-to-one: $\forall (m_1, n_1) \in X$ and $(m_2, n_2) \in X$, if $g \circ f(m_1, n_1) = g \circ f(m_2, n_2)$

$(m_1, n_1) = (m_2, n_2)$

No, there cannot. f's output solely depends on m. By definition of a function, g will always produce the same output for the same input. Since f is the input function in $g \circ f$, nonidentical pairs (m, n_1) and (m, n_2) will produce the same output for $g \circ f$. In other words, $g \circ f(m, n_1) = g \circ f(m, n_2)$, even when $(m, n_1) \neq (m, n_2)$.

d) No, $f(m,n)$ only outputs even integers $2m$. This means half of g's domain (odd #s) $\mathbb{Z}$ is eliminated. Since g is a function and a function can only have one output value associated w/ every input value, at most half of the values in co-domain $\mathbb{Z}$ can have an associated input value (m,n).

∪ = union (or)
∩ = intersect (and)

UID: _ ________

3. (a) Let X be a nonempty set, and $Y \subseteq X$ be a subset of X. Consider the relation R on $\mathcal{P}(X)$ defined by $A\,R\,B$ if and only if $A \cap Y = B \cap Y$. Show that R is an equivalence relation. reflexive, symmetric, transitive

(b) Suppose $X = \{1, 2, 3, 4, 5, 6\}$ and $Y = \{1, 3, 4, 5\}$. Compute the equivalence class of $A = \{1, 2, 3\}$ for the equivalence relation defined in part (a).

(c) Suppose S is an equivalence relation on a nonempty set Z such that S is also a partial order. Show that $S = \{(x, x) \mid x \in Z\}$ (in other words, every element is related to itself but to no other elements).

a) $P(X)$ = all subsets of X. Let $Y \in P(X)$

Reflexive: Is xRx? Yes, $x \cap Y = x \cap Y$ ✓

Symmetric: If xRz, then zRx?

xRz means $x \cap Y = z \cap Y$. Since equality is symmetric, it follows that $z \cap Y = x \cap Y$, which means zRx ✓

Transitive: Let xRy and yRz. Show xRz.

xRz means $x \cap Y = y \cap Y$. yRz means $y \cap Y = z \cap Y$.

Since equality is transitive, $x \cap Y = z \cap Y \Rightarrow xRz$ must be true ✓

b) $[\{1,2,3\}] = \{x \in X \mid xR\{1,2,3\}\}$

= all elements s.t. $A \cap Y = x \cap Y$

Domain is all subsets of X

$A \cap Y = \{1,3\}$. Find all elements x s.t. $\{1,3\} = x \cap Y$

$[\{1,2,3\}] = \{\{1,2,3\}, \{1,3\}, \{1,3,6\}, \{1,2,3,6\}\}$

c) One of the properties of a partial order is that it is antisymmetric.

Antisymmetric: If xRy, and yRx, then $x = y$

An equivalence relation is symmetric, i.e. If xRy, then yRx — this is by symmetric property of equivalence relations

Proof: Since S is an equivalence relation, we have that $\forall x, y \in X$ where $(x,y) \in S$, $(y,x) \in S$. This is due to equiv. relations being symmetric. But since S is a partial order, we also have that S is antisymmetric, meaning if $(x,y) \in S$ and $(y,x) \in S$, then $x = y$. So the only elements that can belong to S are elements of form (x,x). Since S is reflexive, $\forall x \in Z$ $(x,x) \in S$ must be true.

We know S is symmetric because it is an equivalence relation.

We have shown $S = \{(x,x) \mid x \in Z\}$

4. Let X and Y be finite sets, with $|X| = n$, $|Y| = p$.
   (a) How many one-to-one functions are there from X to Y?
   (b) How many reflexive relations are there on X?

   Justify your answers.

a) one-to-one: $\forall x_1, x_2 \in$ domain, if $f(x_1) = f(x_2)$ then $x_1 = x_2$

Every element in domain must have associated w/ it an element in co-domain

There are p choices for the first element in X's corresponding element in Y. There are p-1 choices for the 2nd element, and so on until the nth element, where there are p-n+1 choices

Ans: $\frac{p!}{(p-n)!} = p \cdot (p-1) \cdot \ldots \cdot (p-n+1)$

b) Reflexive xRx exists for all elements in X

n^2 total relations.

Must contain $x_1 R x_1 \ \ldots \ x_n R x_n$

any # of other relations

other pairs: 1st element: n-1

$n \cdot (n-1)$ other pairs, inclusion optional

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