21F-MATH61-2 Final

[redacted]

TOTAL POINTS

60 / 60

QUESTION 1

1 Question 1 6 / 6

✓ **+ 1 pts** (a): Correct (linear with constant coefficients, but not homogeneous)

✓ **+ 1 pts** (b): Base case

✓ **+ 1 pts** (b): Fix $$n\ge1$$ (or equivalent)

✓ **+ 1 pts** (b): Assume $$a_{2n-1}=n^2+1$$ (or equivalent)

✓ **+ 1 pts** (b): Use recurrence relation to write $$a_{2(n+1)-1}$$ in terms of $$a_{2n-1}$$ (or equivalent)

✓ **+ 1 pts** (b): Use induction hypothesis

+ 0 pts (a): Incorrect/missing or no/incorrect explanation

+ 0.5 pts (a): Partially correct explanation

- 1 pts (b): Inductive step done in reverse

QUESTION 2

2 Question 2 7 / 7

✓ **- 0 pts** (a) Correct.

✓ **- 0 pts** (b) Correct.

✓ **- 0 pts** (c) Correct.

✓ **- 0 pts** (d) Correct.

QUESTION 3

3 Question 3 7 / 7

✓ **- 0 pts** Correct

QUESTION 4

4 Question 4 7 / 7

✓ **- 0 pts** Correct

QUESTION 5

5 Question 5 5 / 5

✓ **- 0 pts** Correct

QUESTION 6

6 Question 6 5 / 5

✓ **- 0 pts** (a) Correct.

✓ **- 0 pts** (b) Correct.

✓ **- 0 pts** (c) Correct.

QUESTION 7

7 Question 7 8 / 8

✓ **- 0 pts** Correct

QUESTION 8

8 Question 8 7 / 7

✓ **+ 3 pts** Part a

✓ **+ 2 pts** Part b

✓ **+ 2 pts** Part c

+ 0 pts Incorrect

QUESTION 9

9 Question 9 8 / 8

✓ **+ 1 pts** (a) = True

✓ **+ 1 pts** (b) = False

✓ **+ 1 pts** (c) = False

✓ **+ 1 pts** (d) = True

✓ **+ 1 pts** (e) = True

✓ **+ 1 pts** (f) = True for finite graphs, False for infinite graphs. Since "finite" wasn't specified, both will be counted as correct responses.

✓ **+ 1 pts** (g) = False

✓ **+ 1 pts** (h) = False (it's not even an integer), but there was a mistake in the similar calculation in lecture. Therefore this one will be taken out of the exam and everyone gets the point.

Final Exam

Name: [redacted]

UID: [redacted]

Instructions: Do not open this exam until instructed to do so. You will have 3 hours to complete the exam. Please print your name and student ID number above. You may **not** use calculators, books, notes, or any other material to help you. Please make sure your phone is silenced and stowed where you cannot see it. You may use the front and back of the page for your answers. If you need scratch paper, please ask one of the proctors. You must **show all your work** to receive credit.

Note: In this entire exam, you may leave your numerical responses written as products, fractions, and with factorials. However, you are not allowed to leave expressions of the form $C(n,k)$, $P(n,k)$, etc. As always, you need to justify all your answers.

Question	Points	Score
1	6	
2	7	
3	7	
4	7	
5	5	
6	5	
7	8	
8	7	
9	8	
Total:	60	

UID: [redacted]

1. Consider the sequence a_n, $n \geq 1$, defined by the recurrence relation

$$a_n = a_{n-2} + n, \qquad \text{for } n \geq 3,$$

and with initial conditions $a_1 = 2$, $a_2 = 5$.

(a) (1 point) Is this a linear homogeneous recurrence relation with constant coefficients? If so, what is its order? If not, explain why not.

(b) (5 points) Prove by induction that for every $n \geq 1$: $a_{2n-1} = n^2 + 1$.

a. No, it is not of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

where $c_k \neq 0$

i.e. $n \neq c_k a_{n-k}$ for some k.

$4-1$

$4+1$

$n=2$

$a_3 = 2+2$
$= 4 \neq 5$

b. Basis step: $n=1$

$a_{2(1)-1} = a_1 = 2$

$(1)^2+1 = 2$ ✓

Inductive step: Assume that for some $n \geq 1$

$a_{2n-1} = n^2+1$

Then consider $n+1$,

$a_{2(n+1)-1} = a_{2n+1} = a_{(2n+1)-2} + (2n+1)$

$= a_{2n-1} + 2n + 1$

by assumption

$\Rightarrow = \underbrace{(n^2+1) + 2n + 1}_{(n+1)^2} = (n+1)^2 + 1$ ✓ □

UID: [redacted]

2. Let X, Y be sets, and $f : X \to Y$ be a function.

(a) (1 point) Define what it means for f to be one-to-one.

(b) (1 point) Define what it means for f to be onto.

Let now Z also be a set and $g : X \to Z$ be a function. Define the new function

$$h : X \to Y \times Z : h(x) = (f(x), g(x)).$$

(c) (3 points) Show that if f and g are one-to-one, then h is one-to-one.

(d) (2 points) Prove or give a counterexample: if f and g are onto, then h is onto.

a. f is one-to-one iff for $x_1, x_2 \in X$
if $f(x_1) = f(x_2)$, then $x_1 = x_2$

b. f is onto iff for every $y \in Y$
there exists some $x \in X$ such that
$f(x) = y$.

c. Let $x_1, x_2 \in X$ and $h(x_1) = (f(x_1), g(x_1))$
and $h(x_2) = (f(x_2), g(x_2))$ and $h(x_1) = h(x_2)$.
$\Rightarrow (f(x_1), g(x_1)) = (f(x_2), g(x_2))$
$\Rightarrow f(x_1) = f(x_2)$ and $g(x_1) = g(x_2)$
Since f and g are both one-to-one
$x_1 = x_2$
$\Rightarrow h$ is thus one-to-one.

d. Let $X, Y, Z = \mathbb{Z}$
and $f(x) = x$
$g(x) = x$
both are onto since every $z \in \mathbb{Z}$ has an $x \in \mathbb{Z}, x = z$,
where $f(x) = z$ and $g(x) = z$.
But note the value $(1, 2) \notin h$ since then
$f(x) = 1$ and $g(x) = 2 \Rightarrow x = 1$ and $x = 2$
which is a contradiction $\Rightarrow$ no $h(x)$ s.t. $h(x) = (1, 2) \Rightarrow h$ is not onto
$\Rightarrow$ this is a counter example.

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3. (a) (2 points) Let R be a relation on a set X. Define what it means for R to be reflexive, symmetric, anti-symmetric, and transitive.

Now let X be the set of 6-bit strings. Define a relation R on X by sRt if and only if s and t have the same number of zeroes. (So for instance $011100\,R\,010101$, but $011100\,\not R\,010010$.)

(b) (3 points) Show that R is an equivalence relation.

(c) (2 points) How many elements are there in the equivalence class of 001100? Explain.

a. R is Reflexive $\Leftrightarrow$ for all $x \in X$, xRx

R is symmetric $\Leftrightarrow$ if $x, y \in X$ and xRy then yRx.

R is anti-symmetric $\Leftrightarrow$ if $x, y \in X$ and xRy and yRx then $x = y$.

R is transitive $\Leftrightarrow$ if $x, y, z \in X$ and xRy and yRz then xRz.

b. Reflexive: for any $s \in X$, let s have k zeroes, then s has the same number of zeroes as itself. $\Rightarrow sRs$.

Symmetric: Let $s, y \in X$ and sRy, then if s has k zeroes, y also has k zeroes. Then y has k zeroes, and s has the same number of zeroes as y $\Rightarrow yRs$.

Transitive: let $s, y, z \in X$ and sRy and yRz. Let s have k zeroes. Then since sRy, y also has k zeroes. Since yRz z also has k zeroes. Thus s and z have the same number of zeroes (k). $\Rightarrow sRz$.

c. 001100 has four zeroes, so we can count by choosing any four places in a 6bit string to put zeroes, and leave ones everywhere else

$$\Rightarrow \frac{6!}{2!\,4!} = |[001100]| = |\{s \mid s \in X \text{ and } s \text{ has 4 zeroes}\}|$$

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4. (a) (1 point) How many sequences of length 6 can we form with the numbers from $\{1, 2, \ldots, 9\}$? (Repetition is allowed.)

   (b) (3 points) How many such sequences start with 1 or end with 9?

   (c) (3 points) For how many of the sequences in part (a) do the 6 numbers add up to 15?

$9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9$

a. 9^6, since 9 possible choices per digit.

b. start with 1

$1 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 = 9^5$

end with 9

$9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 1 = 9^5$

both

$1 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 1 = 9^4$

$\Rightarrow \boxed{9^5 + 9^5 - 9^4}$

c. Create a sequence of 6 ones

$\Rightarrow$ has sum of 6.

1 1 1 1 1 1

Then need to distribute 9 (+1)s to the 6 digits

$\Rightarrow C(9+6-1, 6-1) = \frac{14!}{9! \cdot 5!}$

However no digit > 9. Which occurs only when all 9 (+1)s go to the same digit.

$\Rightarrow$ 6 ways

$\Rightarrow \boxed{\frac{14!}{9!\,5!} - 6}$

999

512
−2
510

9+9
12

10

99 2^4

2+9 = 11

9

(10),
1 (10)

29
92

$C(9+1, 1) = 10$

18 − 2

8

29 38
92 83
47 56
74 65

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5. Consider the recurrence relation $a_n = 3a_{n-1} - 2a_{n-2}$ for $n \geq 2$ with initial conditions $a_0 = 0$ and $a_1 = 2$.

   (a) (1 point) Is this a linear homogeneous recurrence relation with constant coefficients? If so, what is its order? If not, explain why not.

   (b) (4 points) Solve the recurrence relation.

a. Yes, with order 2

b. ~~By the theorem~~, since a is a linear homogeneous recurrence relation or order 2, a_n is solvable by $bs^n + dt^n$

where $s, t = k$ where $k^2 - 3k + 2 = 0$

$\Rightarrow (k-2)(k-1) = 0$

$\Rightarrow k = 2, 1$

$\Rightarrow s = 2, t = 1$

Then $0 = b(2)^0 + d(1)^0 = b + d$

and $2 = b(2)^1 + d(1)^1 = 2b + d$

$\Rightarrow d = -2$ and $b = 2$

$\Rightarrow a_n = 2(2^n) - 2(1^n)$

$= \boxed{2 \cdot 2^n - 2}$

$a_2 = 6 - 0$
$= 6$
$8 - 2$ ✓

$a_3 = 18 - 4$
$= 14$
$16 - 2$ ✓

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6. (a) (1 point) State (a version of) the pigeonhole principle.

   (b) (3 points) Let $n \geq 1$ be an integer. Use the pigeonhole principle to show that every $(n+1)$-element subset of $\{1, \dots, 2n\}$ contains two consecutive integers.

   (c) (1 point) Is the same statement still true if we replace "$(n+1)$-element subset" by "n-element subset"? Justify your answer.

a. If there are n pigeons and k pigeonholes and each pigeon flies into a pigeonhole and $n > k$, then there must exist a pigeonhole with at least two pigeons.

b. Let X be the set of some consecutive numbers constructed as $\{(2k-1, 2k) \mid 1 \leq k \leq n\}$. Then $|X| = n$. Every integer in $\{1, \dots, 2n\}$ appears in one element of X. Thus, by the pigeonhole principle, a $(n+1)$-element subset of $\{1, \dots, 2n\}$ has cardinality $n+1 > n$ and there will be at least two elements of the subset that appear in the same element of X. Thus they are two consecutive integers.

c. No, because then $|X| = n$ (→ from (b)) and the subset has n elements, thus pigeonhole does not apply. Namely if the subset is all the odd integers it will have n elements but no consecutive integers.

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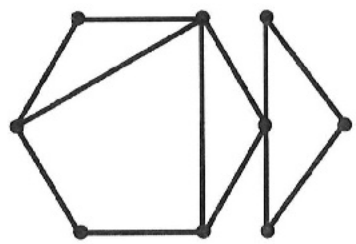
7. Let $G = (V, E)$ be a simple undirected graph.
   (a) (2 points) Define *Euler cycle* and *Hamiltonian cycle.*
   (b) (2 points) Show (without using theorems from class) that if G has a vertex of odd degree, then G does not have an Euler cycle.
   (c) (2 points) Can you add one edge to the following graph to make it into a simple undirected graph that has an Euler cycle? Explain. (You may use theorems from class here.)



   (d) (2 points) Does the graph given in part (c) have a Hamiltonian cycle? Explain.

a. An Euler cycle is a cycle (in a simple connected graph G) that visits every vertex and visits every edge exactly once (except for beginning and end)

A Hamiltonian cycle is a cycle (in a connected simple graph G) that visits every vertex exactly one (except for the beginning and end).

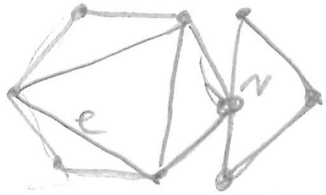
b. G has a vertex of odd degree ⇒ G does not have an Euler cycle.

let $v \in V$ if $G=(V,E)$ and $\deg(v) = 2n+1$ for some $n \in \mathbb{Z}_{\geq 0}$.

If v is part of the Euler cycle (and not the start of the cycle), then the cycle can be written $(\ldots, v_i, v, v_j, \ldots, v_k, v, v_l, \ldots)$ (some amount of repetitions) for some repetition of v. Then every time v appears in the cycle, one edge is visited to reach v, then another is used to leave v. Then the number of edges used to visit and leave v must be $2k$ for every visit of v. Let $k=n$, the there is only one edge left incident on v. But Euler cycle requires all edges be visited. Visiting this edge brings us back to v, but now there is no way to leave v and continue the cycle. v also cannot be the start vertex of the cycle because by similar logic, everytime we leave v by an edge, another edge is used to arrive back in v to complete the cycle, thus still leaving an edge unvisited.

⇒ No Euler cycle

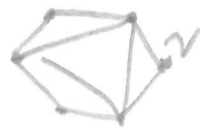
c.



Added e

All vertices have ~~degree~~ even degree ⇒ then exists an Euler cycle

d. No, because ~~vertex~~ labeled vertex v can only be visited once by the cycle, however the cycle needs to cross it each time to cross between subgraph A



and subgraph B

and then return to the starting vertex back on the other subgraph. Thus v is visited at least twice ⇒ Not a Hamiltonian cycle.

If v is the starting vertex, then v is visited at the beginning and another time when crossing between the subgraphs, but then the cycle is complete, yet the other subgraph has not be completed visited ⇒ Not a Hamiltonian cycle

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8. Let $G = (V, E)$ be an undirected connected planar graph with at least one cycle. Denote by v the number of vertices, e the number of edges, and f the number of faces of G. Assume that every cycle of G has length exactly 5.

   (a) (3 points) Explain why $2e \geq 5f$.

   (b) (2 points) State Euler's formula, and use it show that $5v - 3e \geq 2$.

   (c) (2 points) Does there exist such a graph G that is bipartite? If yes, draw an example. If not, explain why not.

c. For every edge it divides at most 2 faces (
But since every cycle has length exactly 5,
each face is bounded by at least 5 edges.
Thus $2e \geq 5f$.

b. Euler's formula: $f = e - v + 2$
since $2e \geq 5f \Rightarrow f \leq \frac{2}{5}e$
$\Rightarrow \frac{2}{5}e \geq e - v + 2$
$\Rightarrow -\frac{3}{5}e + v \geq 2 \Rightarrow 5v - 3e \geq 2$

c. No, since every cycle has length 5.
Assume G is bipartite, for contradiction.
Let c be a cycle $(v_1, v_2, v_3, v_4, v_5, v_1)$ of length 5.
Since v_1 is incident on v_2, $v_1 \in V_1$ and $v_2 \in V_2$ where V_1 and V_2 are the bipartite partitions of G. (i.e. $V_1 \cap V_2 = \emptyset$ and no $v_1 \in V_1$ is adjacent nor any $v_2 \in V_2$ is adjacent). Similarly $v_3 \in V_1$, $v_4 \in V_2$, $v_5 \in V_1$. But then both $v_1, v_5 \in V_1$.
Yet in cycle c, there is an edge from v_5 to v_1
which is a contradiction

9. (8 points) For each of the following statements, write TRUE or FALSE (please not just T or F). **No justification is required.**

   (a) For every $n > 1$, there are exactly $2^{(n^2)}$ relations on a set of size n. P(n×n)

   TRUE

   (b) Let R be the relation on $\mathbb{Z}$ defined by xRy if and only if at least one of x and y is prime. Then R is transitive. 4R7 7R6 4R6

   FALSE

   (c) For every $n \geq k \geq 1$: $C(n,k) = P(n,k) \cdot k!$ $\frac{n!}{k!(n-k)!} \neq \frac{n!}{(n-k)!} k!$

   FALSE

   (d) There are $C(10,5)C(5,3)$ strings of length 10 that contain exactly 5 'a's, 3 'b's and 2 'c's.

   TRUE

   (e) For every $n \geq 1$: $\sum_{k=0}^{n} C(n,k) = 2^n$. $C(1,0)+C(1,1) = 2^1$ ✓ $C(2,0)+C(2,1)+C(2,2) = 4$ ✓

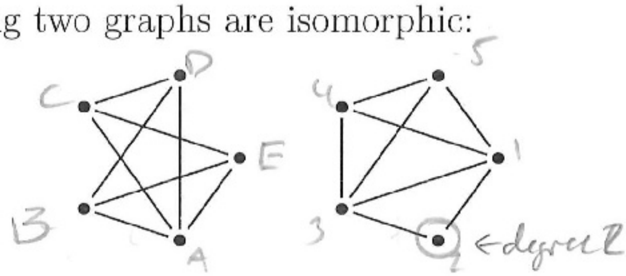
   TRUE

   (f) Let $G = (V,E)$ be a simple undirected graph. If every vertex has degree at least 2, then G contains a cycle.

   FALSE

   (g) The following two graphs are isomorphic:

   . A

   FALSE

   (h) There are exactly $\dfrac{C(5,2)^3}{6}$ undirected graphs with vertex set $\{1,2,3,4,5\}$, that have 3 edges, and that have no loops.

   C(5,2)

   TRUE