# 21F-MATH61-2 Final

QUAN DO

TOTAL POINTS

**54 / 60**

QUESTION 1

## 1 Question 1 **5 / 6**

✓ **+ 1 pts** (a): Correct (linear with constant coefficients, but not homogeneous)

✓ **+ 1 pts** (b): Base case

**+ 1 pts** (b): Fix $$n\ge1$$ (or equivalent)

✓ **+ 1 pts** (b): Assume $$a_{2n-1}=n^2+1$$ (or equivalent)

✓ **+ 1 pts** (b): Use recurrence relation to write $$a_{2(n+1)-1}$$ in terms of $$a_{2n-1}$$ (or equivalent)

✓ **+ 1 pts** (b): Use induction hypothesis

**+ 0 pts** (a): Incorrect/missing or no/incorrect explanation

**+ 0.5 pts** (a): Partially correct explanation

**- 1 pts** (b): Inductive step done in reverse

1 You need to say what $$n$$ is.

QUESTION 2

## 2 Question 2 **7 / 7**

✓ **- 0 pts** (a) Correct.

✓ **- 0 pts** (b) Correct.

✓ **- 0 pts** (c) Correct.

✓ **- 0 pts** (d) Correct.

QUESTION 3

## 3 Question 3 **6 / 7**

✓ **- 1 pts** c) Didn't account for the double counting (i.e. got 30 instead of 30/2=15)

2 The $$x_1\ne x_2$$ is unnecessary.

QUESTION 4

## 4 Question 4 **5 / 7**

✓ **- 2 pts** c) Properly accounted for the fact that the numbers must be at most 9, namely, by subtracting 6, but did not get the rest correct.

QUESTION 5

## 5 Question 5 **5 / 5**

✓ **- 0 pts** Correct

QUESTION 6

## 6 Question 6 **3 / 5**

✓ **- 0 pts** (a) Correct.

✓ **- 0.5 pts** (c) Correctly states no, but issue with counterexample.

✓ **- 1.5 pts** (b) Some correct idea, but issues with applying the pigeonhole principle.

3 You need a concrete counterexample. Otherwise there may be a different proof that could work for $$n$$ elements.

4 ??

5 How does this follow from the construction of $$Y$$ and $$Z$$?

QUESTION 7

## 7 Question 7 **8 / 8**

✓ **- 0 pts** Correct

QUESTION 8

## 8 Question 8 **7 / 7**

✓ **+ 3 pts** Part a

✓ **+ 2 pts** Part b

✓ **+ 2 pts** Part c

**+ 0 pts** Incorrect

QUESTION 9

## 9 Question 9 **8 / 8**

✓ **+ 1 pts** (a) = True

✓ + 1 pts (b) = False

✓ + 1 pts (c) = False

✓ + 1 pts (d) = True

✓ + 1 pts (e) = True

✓ + 1 pts (f) = True for finite graphs, False for infinite graphs. Since "finite" wasn't specified, both will be counted as correct responses.

✓ + 1 pts (g) = False

✓ + 1 pts (h) = False (it's not even an integer), but there was a mistake in the similar calculation in lecture. Therefore this one will be taken out of the exam and everyone gets the point.

# Final Exam

Name: Quan Do

UID: [redacted]

**Instructions:** Do not open this exam until instructed to do so. You will have 3 hours to complete the exam. Please print your name and student ID number above. You may **not** use calculators, books, notes, or any other material to help you. Please make sure your phone is silenced and stowed where you cannot see it. You may use the front and back of the page for your answers. If you need scratch paper, please ask one of the proctors. You must **show all your work** to receive credit.

**Note:** In this entire exam, you may leave your numerical responses written as products, fractions, and with factorials. However, you are not allowed to leave expressions of the form $C(n, k)$, $P(n, k)$, etc. As always, you need to justify all your answers.

| Question | Points | Score |
|---|---|---|
| 1 | 6 | |
| 2 | 7 | |
| 3 | 7 | |
| 4 | 7 | |
| 5 | 5 | |
| 6 | 5 | |
| 7 | 8 | |
| 8 | 7 | |
| 9 | 8 | |
| Total: | 60 | |

UID: [redacted]

1. Consider the sequence $a_n$, $n \geq 1$, defined by the recurrence relation

$$a_n = a_{n-2} + n, \qquad \text{for } n \geq 3,$$

and with initial conditions $a_1 = 2$, $a_2 = 5$.

(a) (1 point) Is this a linear homogeneous recurrence relation with constant coefficients? If so, what is its order? If not, explain why not.

(b) (5 points) Prove by induction that for every $n \geq 1$: $\overset{a_1,\, a_3,\, a_5}{a_{2n-1}} = n^2 + 1$.

a) No, because the "$n$" term is not in the form $c\,a_{n-k}$, where $c$ is a constant.

b) Basis: $n=1$: $a_{2(1)-1} = a_1 = 2 = (1)^2+1$ ✓

Ind.: Assume $a_{2n-1} = n^2+1$ is true. Prove $a_{2(n+1)-1} = (n+1)^2+1$ for $n \geq 1$.

$a_{2n+2-1} = a_{2n+1} = a_{2n-1} + (2n+1)$ by rec. rel. because $n \geq 1$ so $2n+1 \geq 3$

$= n^2+1 + 2n+1$

$= (n^2+2n+1)+1$

$= (n+1)^2+1$ ✓

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2. Let $X, Y$ be sets, and $f : X \to Y$ be a function.

   (a) (1 point) Define what it means for $f$ to be one-to-one. injective

   (b) (1 point) Define what it means for $f$ to be onto. surjective

   Let now $Z$ also be a set and $g : X \to Z$ be a function. Define the new function

$$h : X \to Y \times Z : h(x) = (f(x), g(x)).$$

   (c) (3 points) Show that if $f$ and $g$ are one-to-one, then $h$ is one-to-one.

   (d) (2 points) Prove or give a counterexample: if $f$ and $g$ are onto, then $h$ is onto.

a) $\forall x_1, x_2 \in X : f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

b) $\forall y \in Y : \exists x \in X$ s.t. $f(x) = y$

c) Assume: $\forall x_1, x_2 \in X : f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ and $g(x_1) = g(x_2) \Rightarrow x_1 = x_2$

Let $x_1, x_2 \in X$. Assume $h(x_1) = h(x_2)$.

$\rightarrow (f(x_1), g(x_1)) = (f(x_2), g(x_2))$

$\Rightarrow f(x_1) = f(x_2)$ and $g(x_1) = g(x_2)$

$\Rightarrow x_1 = x_2$ and $x_1 = x_2$

$h$ is one-to-one because $h(x_1) = h(x_2) \Rightarrow x_1 = x_2$.

d) Let $f(x) = x$, $g(x) = 2x$. $(f, g : \mathbb{R} \to \mathbb{R})$

Then $h(x) = (x, 2x)$. $(h : \mathbb{R} \to \mathbb{R}^2)$

Consider $y = (0, 1)$, $y \in \mathbb{R}^2$

If $h$ is onto, then $\exists x \in \mathbb{R} : h(x) = (0,1)$

$\rightarrow (x, 2x) = (0, 1)$

$\cancel{x = 0}$

$\cancel{2x = 1}$ $\rightarrow$ Counterexample

So $f$ and $g$ are onto $\not\Rightarrow$ $h$ is onto.

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3. (a) (2 points) Let $R$ be a relation on a set $X$. Define what it means for $R$ to be reflexive, symmetric, anti-symmetric, and transitive.

Now let $X$ be the set of 6-bit strings. Define a relation $R$ on $X$ by $sRt$ if and only if $s$ and $t$ have the same number of zeroes. (So for instance $011100\,R\,010101$, but $011100\,\not R\,010010$.)

(b) (3 points) Show that $R$ is an equivalence relation.

(c) (2 points) How many elements are there in the equivalence class of 001100? Explain.

a) Refl: $\forall x \in X : x R x$.

Sym: $\forall x_1, x_2 \in X, x_1 \neq x_2 : x_1 R x_2 \Rightarrow x_2 R x_1$.

Antisym: $\forall x_1, x_2 \in X$, $x_1 R x_2$ and $x_2 R x_1 \Rightarrow x_1 = x_2$.

Trans: $\forall x, y, z \in X$, $x R y$ and $y R z \Rightarrow x R z$.

b) Refl: A string has the same number of zeros as itself, so it must relate to itself.

Sym: Consider 2 distinct strings $x, y$, and $xRy$. This means $x$ and $y$ have the same number of zeroes, so then $yRx$.

Trans: $xRy$ and $yRz$. Suppose $x$ has $n$ zeroes. $xRy \Rightarrow y$ has $n$ zeroes. $yRz \Rightarrow z$ has $n$ zeroes. $X$ and $z$ both have $n$ zeroes, so $xRz$.

$R$ is reflexive, symmetric, and transitive, so it's an equivalence relation.

c) $\Leftrightarrow$ How many strings are there with 4 zeroes?

Consider 6 slots for bits: _ _ _ _ _ _.

Pick a slot for the first "1": 6 options.

———— " ———— second "1": 5 options.

The rest must be zeroes, so there are $6 \cdot 5 = \boxed{30}$ distinct strings/elements in this equivalence class.

4. (a) (1 point) How many sequences of length 6 can we form with the numbers from $\{1, 2, \ldots, 9\}$? (Repetition is allowed.)

   (b) (3 points) How many such sequences start with 1 or end with 9?

   (c) (3 points) For how many of the sequences in part (a) do the 6 numbers add up to 15?

a) Each digit has 9 options: $\boxed{9^6}$

b) Start with 1 : $9^5$
End with 9 : $9^5$
Both : $9^4$

$\boxed{2 \cdot 9^5 - 9^4}$

c) | * | * | * | * | * | * |

* * * * * * * * *

Choose where the remaining 9 stars go out of 6 options: $6^9$
But we can't have all 9 go into one, so exclude 6 choices
The stars are identical, so divide by number of permutations: 9!

$$\boxed{\frac{6^9}{9!} - 6}$$

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5. Consider the recurrence relation $a_n = 3a_{n-1} - 2a_{n-2}$ for $n \geq 2$ with initial conditions $a_0 = 0$ and $a_1 = 2$.

$0, 2, 6, 14, \ldots$

$0 \quad 1 \quad 2 \quad 3$

(a) (1 point) Is this a linear homogeneous recurrence relation with constant coefficients? If so, what is its order? If not, explain why not.

(b) (4 points) Solve the recurrence relation.

a) Yes, 2nd order.

b) $c_1 = 3,\ c_2 = -2$

$t^2 = 3t - 2$

$t^2 - 3t + 2 = 0$

$(t-2)(t-1) = 0 \rightarrow t = 1, 2 \Rightarrow b(1)^n + d(2)^n$ is a solution

$n=0$: $b(1)^0 + d(2)^0 = 0$

$b + d = 0$

$b = -d$

$n=1$: $b(1)^1 + d(2)^1 = 2$

$b + 2d = 2$

$-d + 2d = 2$

$d = 2$ and $b = -2$

$a_n = (-2)1^n + 2(2)^n$

$= \boxed{2^{n+1} - 2}$

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6. (a) (1 point) State (a version of) the pigeonhole principle.

   (b) (3 points) Let $n \geq 1$ be an integer. Use the pigeonhole principle to show that every $(n+1)$-element subset of $\{1, \ldots, 2n\}$ contains two consecutive integers.

   (c) (1 point) Is the same statement still true if we replace "$(n+1)$-element subset" by "$n$-element subset"? Justify your answer.

a) Consider $f: X \to Y$, $|X| = n$, $|Y| = m$.
If $n > m$, then there are at least 2 distinct $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$.

b) Let $X = \{1, \ldots, 2n\}$. When constructing $Y \subseteq X$ s.t. $|Y| = n+1$, we choose $n+1$ elements from $X$. Suppose $Y$ has no consecutive integers. Then when we choose an element $x \in X$ to go in $Y$, we can also choose $x+1$ to go into a new set $Z$. For each $x$ to not be consecutive, we restrict $x \notin Z$ and $x+1 \notin Z$. If $x = 2n$, then add $2n+1$ to $X$, so $|X| = 2n+1$. For each element chosen, we have decreased the remaining elements in $X$ by 2, so we need at least $2(n+1) = 2n+2$ elements in $X$. $|X| < 2n+2$, so it is impossible to have no consecutive integers.

c) In this case, we need $2n$ elements in $X$. $|X| \geq 2n$, so the statement is no longer necesarily true.

4

5

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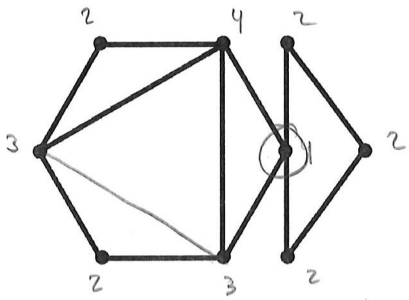
7. Let $G = (V, E)$ be a simple undirected graph.

(a) (2 points) Define *Euler cycle* and *Hamiltonian cycle.*

(b) (2 points) Show (without using theorems from class) that if $G$ has a vertex of odd degree, then $G$ does not have an Euler cycle.

(c) (2 points) Can you add one edge to the following graph to make it into a simple undirected graph that has an Euler cycle? Explain. (You may use theorems from class here.)




(d) (2 points) Does the graph given in part (c) have a Hamiltonian cycle? Explain.

a) An Euler cycle is a cycle that contains every vertex and passes through every edge exactly once.

A Hamiltonian cycle is a cycle that passes through every vertex exactly once.

b) Suppose $G = (V, E)$, and $v \in V$, $\delta(v)$ is odd.

Consider an Euler cycle that passes through $v$.

If $v$ is the start/end of the cycle:

The cycle must leave/enter $v$ through 2 edges. Adding these two edges does not change whether $\delta(v)$ is even or odd, so ignore them.

Suppose the cycle passes through $v$ $n$ times. In each of these times, there is an edge before $v$ in the path, and after $v$ (all edges in the path must be unique). So the path at this point looks like:

$$(\ldots, e_1, v, e_2, \ldots).$$

This means there must be exactly 2 edges incident on $v$ for every time the cycle passes through $v$, so there must be $2n$ edges. But we know that $\delta(v)$ must be odd. This is a contradiction, so the cycle cannot be an Euler cycle.

c) There are 2 vertices with odd degree. Adding an edge between them makes their degrees even. Now all the vertices have an even degree, so there is now an Euler cycle.

d) No. Let the vertex circled in the drawing be $v$, the set of vertices to the left of $v$ be $V_1$, and the set of all vertices to the right of $v$ be $V_2$. Let's try to construct a Hamiltonian cycle. Pick a starting vertex $v_0$.

If $v_0 \in V_1$, the cycle must pass through $v$ to get to the vertices in $V_2$, then pass through $v$ again to get back to $v_0$.

If $v_0 \in V_2$, the cycle must pass through $v$ to get to the vertices in $V_1$, then pass through $v$ again to get back to $v_0$.

If $v_0 = v$, the next vertex $v'$ can be in $V_1$ or $V_2$.

If $v' \in V_1$, then the cycle must pass through $v$ in the middle of the path to get to the vertices in $V_2$.

If $v' \in V_2$, then the cycle must pass through $v$ in the middle of the path to get to the vertices in $V_1$.

No matter how we construct the cycle, we must pass through $v$ at least twice, so the cycle cannot be Hamiltonian.

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8. Let $G = (V, E)$ be an undirected connected planar graph with at least one cycle. Denote by $v$ the number of vertices, $e$ the number of edges, and $f$ the number of faces of $G$. Assume that every cycle of $G$ has length exactly 5.

   (a) (3 points) Explain why $2e \geq 5f$.

   (b) (2 points) State Euler's formula, and use it show that $5v - 3e \geq 2$.

   (c) (2 points) Does there exist such a graph $G$ that is bipartite? If yes, draw an example. If not, explain why not.

a) It takes exactly 5 edges to form a face because every cycle has length 5. Each of these edges contribute to 2 faces. If all edges are in a cycle, then $2e = 5f$, but there may be extra edges, so $2e \geq 5f$.

b) $f = e - v + 2$

$5f = 5(e-v+2) \leq 2e$

$3e - 5v + 10 \leq 0$

$5v - 3e \geq 10 > 2$

$\Rightarrow 5v - 3e \geq 2$ ✓

c) No. In a bipartite graph, every cycle must have the form

$$(v_0, v_0', v_1, v_1', \dots, v_n, v_n', v_0),$$

where $v_i \in V_1$ and $v_i' \in V_2$, with $\{V_1, V_2\}$ being the bipartite partition. This is because each edge has to be between a vertex in $V_1$ and a vertex in $V_2$. This means the cycle has length $2(n+1)$, which is even. $G$ has a cycle of length 5, which is not even, so $G$ is not bipartite.

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9. (8 points) For each of the following statements, write TRUE or FALSE (please not just T or F). **No justification is required.**

(a) For every $n > 1$, there are exactly $2^{(n^2)}$ relations on a set of size $n$.

True

(b) Let $R$ be the relation on $\mathbb{Z}$ defined by $xRy$ if and only if at least one of $x$ and $y$ is prime. Then $R$ is transitive.

$4R5,\ 5R6,\ 4 \not R 6$

False

(c) For every $n \geq k \geq 1$: $C(n,k) = P(n,k) \cdot k!$

$C(n,k) = \frac{n!}{(n-k)!k!}$ $\quad P(n,k)\cdot k! = \frac{n!}{(n-k)!} \cdot k!$ $\quad$ False

(d) There are $C(10,5)C(5,3)$ strings of length 10 that contain exactly 5 'a's, 3 'b's and 2 'c's.

True

c c a a b a a a b b

Choose 5 'a's : $\binom{10}{5}$

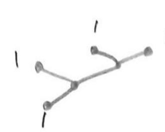
Choose 3 'b's : $\binom{5}{3}$

(e) For every $n \geq 1$: $\sum_{k=0}^{n} C(n,k) = 2^n$.

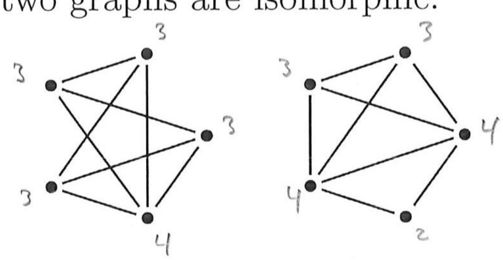
$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$ $\quad a=b=1: (1+1)^n = 2^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^k = \sum_{k=0}^{n} C(n,k)$ $\quad$ True

(f) Let $G = (V,E)$ be a simple undirected graph. If every vertex has degree at least 2, then $G$ contains a cycle.

True



(g) The following two graphs are isomorphic:

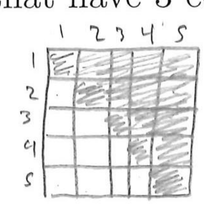



False

(h) There are exactly $\frac{C(5,2)^3}{6}$ undirected graphs with vertex set $\{1,2,3,4,5\}$, that have 3 edges, and that have no loops.

$\binom{5}{2} = \frac{5\cdot 4}{2} = 10$

$10^3 = 1000$




10 free squares, choose 3

$\binom{10}{3} = \frac{10!}{7!\,3!} = \frac{10\cdot 9^3 \cdot 8^4}{\cancel{3}\cdot\cancel{2}} = 120$

$\frac{1000}{6} \notin \mathbb{Z}$

False