21F-MATH61-2 Final

MATTHEW FIORELLA

TOTAL POINTS

58.5 / 60

QUESTION 1

1 Question 1 **5 / 6**

✓ + **1 pts** **(a): Correct (linear with constant coefficients, but not homogeneous)**

✓ + **1 pts** **(b): Base case**

+ **1 pts** (b): Fix $$n\ge1$$ (or equivalent)

✓ + **1 pts** **(b): Assume $$a_{2n-1}=n^2+1$$ (or equivalent)**

✓ + **1 pts** **(b): Use recurrence relation to write $$a_{2(n+1)-1}$$ in terms of $$a_{2n-1}$$ (or equivalent)**

✓ + **1 pts** **(b): Use induction hypothesis**

+ **0 pts** (a): Incorrect/missing or no/incorrect explanation

+ **0.5 pts** (a): Partially correct explanation

- **1 pts** (b): Inductive step done in reverse

(1) You need to say what $$n$$ is.

QUESTION 2

2 Question 2 **7 / 7**

✓ - **0 pts** **(a) Correct.**

✓ - **0 pts** **(b) Correct.**

✓ - **0 pts** **(c) Correct.**

✓ - **0 pts** **(d) Correct.**

QUESTION 3

3 Question 3 **7 / 7**

✓ - **0 pts** **Correct**

QUESTION 4

4 Question 4 **7 / 7**

✓ - **0 pts** **Correct**

QUESTION 5

5 Question 5 **5 / 5**

✓ - **0 pts** **Correct**

QUESTION 6

6 Question 6 **5 / 5**

✓ - **0 pts** **(a) Correct.**

✓ - **0 pts** **(b) Correct.**

✓ - **0 pts** **(c) Correct.**

QUESTION 7

7 Question 7 **7.5 / 8**

✓ - **0.5 pts** **(d): Correct, but argument is not quite complete**

(2) As defined in this class, an Euler cycle must also pass through every vertex. Other sources differ on this.

(3) Why?

QUESTION 8

8 Question 8 **7 / 7**

✓ + **3 pts** **Part a**

✓ + **2 pts** **Part b**

✓ + **2 pts** **Part c**

+ **0 pts** Incorrect

QUESTION 9

9 Question 9 **8 / 8**

✓ + **1 pts** **(a) = True**

✓ + **1 pts** **(b) = False**

✓ + **1 pts** **(c) = False**

✓ + **1 pts** **(d) = True**

✓ + **1 pts** **(e) = True**

✓ + **1 pts** **(f) = True for finite graphs, False for infinite graphs. Since "finite" wasn't specified, both will be counted as correct responses.**

✓ + **1 pts** **(g) = False**

✓ **+ 1 pts** (h) = **False (it's not even an integer), but there was a mistake in the similar calculation in lecture. Therefore this one will be taken out of the exam and everyone gets the point.**

Final Exam

Name: Matthew Fiorella

UID: 605 593 729

Instructions: Do not open this exam until instructed to do so. You will have 3 hours to complete the exam. Please print your name and student ID number above. You may **not** use calculators, books, notes, or any other material to help you. Please make sure your phone is silenced and stowed where you cannot see it. You may use the front and back of the page for your answers. If you need scratch paper, please ask one of the proctors. You must **show all your work** to receive credit.

Note: In this entire exam, you may leave your numerical responses written as products, fractions, and with factorials. However, you are not allowed to leave expressions of the form $C(n, k)$, $P(n, k)$, etc. As always, you need to justify all your answers.

Question	Points	Score
1	6	
2	7	
3	7	
4	7	
5	5	
6	5	
7	8	
8	7	
9	8	
Total:	60	

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1. Consider the sequence a_n, $n \geq 1$, defined by the recurrence relation

$$a_n = a_{n-2} + n, \qquad \text{for } n \geq 3,$$

and with initial conditions $a_1 = 2$, $a_2 = 5$.

(a) (1 point) Is this a linear homogeneous recurrence relation with constant coefficients? If so, what is its order? If not, explain why not.

(b) (5 points) Prove by induction that for every $n \geq 1$: $a_{2n-1} = n^2 + 1$.

(a) This is not a linear homogeneous recurrence relation with constant coefficients because the n term is not permitted in a linear homogeneous recurrence relation with constant coefficients.

(b) Base case: n=1,

$n=1$: $a_{2(1)-1} = (1)^2+1$

from initial conditions: $a_1 = 2$

$2=2$ ✓

Inductive step

Assume $a_{2k-1} = k^2+1$ for $1 \leq k \leq n$

Need to show $a_{2(n+1)-1} = (n+1)^2+1$

We know $a_{2(n+1)-1} = a_{2n+1} = a_{(2n+1)-2} + 2n+1$

So, $a_{2n+1} = a_{2n-1} + 2n+1$

By assumption, we have $a_{2n-1} = n^2+1$

So, $a_{2n+1} = n^2+1+2n+1$

Rearranging terms, we get $a_{2n+1} = n^2+2n+1+1$

Which yields $a_{2(n+1)-1} = (n+1)^2+1$

This proves the inductive step ✓

∴ for every $n \geq 1$: $a_{2n-1} = n^2+1$. ✓

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2. Let X, Y be sets, and $f : X \to Y$ be a function.

   (a) (1 point) Define what it means for f to be one-to-one.

   (b) (1 point) Define what it means for f to be onto.

   Let now Z also be a set and $g : X \to Z$ be a function. Define the new function

$$h : X \to Y \times Z : h(x) = (f(x), g(x)).$$

   (c) (3 points) Show that if f and g are one-to-one, then h is one-to-one.

   (d) (2 points) Prove or give a counterexample: if f and g are onto, then h is onto.

(a) f is one-to-one if $\forall x_1, x_2 \in X$: if $f(x_1) = f(x_2)$, then $x_1 = x_2$

(b) f is onto if $\forall y \in Y : \exists x \in X$ such that $f(x) = y$

(c) Assume for some $x_1, x_2 \in X$, we have $h(x_1) = h(x_2)$

Then we have $(f(x_1), g(x_1)) = (f(x_2), g(x_2))$

Which means $f(x_1) = f(x_2)$ and $g(x_1) = g(x_2)$

We know f and g are one to one, so we have $x_1 = x_2$

Thus if $h(x_1) = h(x_2)$, then $x_1 = x_2$

This satisfies the definition of one-to-one $\therefore$ h is one-to-one.

(d) The statement is false.

Counterexample: let $f : \mathbb{R} \to \mathbb{R} : f(x) = x$, let $g : \mathbb{R} \to \mathbb{R} : g(x) = x$

Note, Both of the functions are onto because $\forall y \in \mathbb{R}$ (codomain), we have that $x = y$, so $\exists x \in \mathbb{R}$ (domain) such that $f(x) = y$ and $g(x) = y$.

So we have $h : \mathbb{R} \to \mathbb{R} \times \mathbb{R} : h(x) = (f(x), g(x))$

$\Rightarrow h(x) = (x, x)$

Check if $\exists x$ s.t. $h(x) = (1, 2)$

$\Rightarrow h(x) = (1, 2)$

$\Rightarrow (x, x) \overset{?}{=} (1, 2)$

$\Rightarrow$ ~~$x = 1, x = 2$~~ F this is impossible, so $h(x) \neq (1, 2)$ $\therefore$ $h(x)$ is not onto,

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3. (a) (2 points) Let R be a relation on a set X. Define what it means for R to be reflexive, symmetric, anti-symmetric, and transitive.

Now let X be the set of 6-bit strings. Define a relation R on X by sRt if and only if s and t have the same number of zeroes. (So for instance $011100\, R\, 010101$, but $011100 \not R\, 010010$.)

(b) (3 points) Show that R is an equivalence relation.

(c) (2 points) How many elements are there in the equivalence class of 001100? Explain.

(a) reflexive: R is reflexive if $\forall x \in X$, xRx

symmetric: R is symmetric if $\forall x,y \in X$, if xRy then yRx.

anti-symmetric: R is antisymmetric if $\forall x,y \in X$, if xRy and yRx, then $x=y$

transitive: R is transitive if $\forall x,y,z \in X$, if xRy and yRz then xRz.

(b) To show R is an equivalence relation, we need to show it is reflexive, symmetric, and transitive.

reflexive?:

~~Assum~~ every 6-bit string will have the same # of 0's as itself. ∴ R is reflexive

symmetric?:

Assume for some 6-bit strings s and t, sRt. Then s and t have the same number of 0's. So we also have tRs. ∴ R is ~~reflexive~~ symmetric

transitive?:

Assume for some six-bit strings a, b, c, aRb and ~~and~~ bRc. Then a and b have the same number of 0's and b and c have the same number of 0's. It follows that a and c have the same number of 0's. So we have aRc ∴ R is transitive

R is ~~transit~~ reflexive, symmetric, and transitive ∴ it is ~~the~~ an equivalence relation □

(c) The number of elements in the equivalence class of 001100 will be all six-bit strings with four 0's as these are all the elements related to 001100. So, we need to choose 4 out of the 6 spots to place zeros. This is just $C(6,4) =$ ~~$\frac{6!}{4!2!}$ elements~~ $\frac{6!}{4!2!}$ = **15 elements**

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4. (a) (1 point) How many sequences of length 6 can we form with the numbers from $\{1, 2, \ldots, 9\}$? (Repetition is allowed.)

   (b) (3 points) How many such sequences start with 1 or end with 9?

   (c) (3 points) For how many of the sequences in part (a) do the 6 numbers add up to 15?

~~(a) $C(6+9-1, 9-1) = C(14,8) = \frac{14!}{8!\,6!}$ sequences~~

~~(b) Inclusion-Exclusion: $|A \cup B| = |A| + |B| - |A \cap B|$~~
~~# sequences that start with 1: 1 _ _ _ _ _ ⇒ C(5+9-1,~~

(a) _ _ _ _ _ _ (9 pcs. 9 pcs. 9 pcs. 9 pcs. 9 pcs. 9 pcs.)

$9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 = 9^6$ sequences

(b) Inclusion-Exclusion: $|A \cup B| = |A| + |B| - |A \cap B|$

# of sequences that start with 1: 1 _ _ _ _ _ ⇒ $9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 = 9^5$ sequences

# of sequences that end with 9: _ _ _ _ _ 9 ⇒ $9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 = 9^5$ sequences

# of sequences that start and end with 1 and 9 respectively: 1 _ _ _ _ 9 ⇒ $9 \cdot 9 \cdot 9 \cdot 9 = 9^4$ sequences

# of sequences that start with 1 or end with 9 $= 9^5 + 9^5 - 9^4$

$9^5 + 9^5 - 9^4$ sequences

(c) ~~$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 15$~~

$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 15$

Let $k_i a_i = x_i$

need $k_i \geq 1$

$a_1\ a_2\ a_3\ a_4\ a_5\ a_6 \{$ _ _ _ _ _ _ _ _ _ $\}$

5 barriers, 9 spots

a_1, a_2, a_3, a_4, a_5

$C(9+5, 5) = C(14, 5)$

~~$\frac{14!}{9!\,5!}$ sequences~~

$C(14,5)$ Need to subtract off seq. where one is 10. That can happen 6 ways

$C(14,5) - 6 = \frac{14!}{9!\,5!} - 6$ sequences

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5. Consider the recurrence relation $a_n = 3a_{n-1} - 2a_{n-2}$ for $n \geq 2$ with initial conditions $a_0 = 0$ and $a_1 = 2$.

   (a) (1 point) Is this a linear homogeneous recurrence relation with constant coefficients? If so, what is its order? If not, explain why not.

   (b) (4 points) Solve the recurrence relation.

(a) This is a linear homogeneous recurrence relation with constant coefficients of order 2.

(b) $a_n = 3a_{n-1} - 2a_{n-2}$

$t^2 - 3t + 2 = 0$

$(t-1)(t-2) = 0$

$t_1 = 1, t_2 = 2$

$a_n = bt_1^n + dt_2^n \Rightarrow a_n = b1^n + d2^n \Rightarrow a_n = b + d2^n$

Plug initial conditions: $a_0 = b + d2^0 \Rightarrow a_0 = b + d \Rightarrow 0 = b + d$

$a_1 = b + d2^1 \Rightarrow a_1 = b + 2d \Rightarrow 2 = b + 2d$

$b + d = 0$, $b + 2d = 2 \Rightarrow d = 2$, $b + 2 = 0 \Rightarrow b = -2$

$a_n = -2 + 2(2)^n$ for $n \geq 0$

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6. (a) (1 point) State (a version of) the pigeonhole principle.

   (b) (3 points) Let $n \geq 1$ be an integer. Use the pigeonhole principle to show that every $(n+1)$-element subset of $\{1, \ldots, 2n\}$ contains two consecutive integers.

   (c) (1 point) Is the same statement still true if we replace "$(n+1)$-element subset" by "n-element subset"? Justify your answer.

(a) If we have n+1 pigeons to put into n pigeonholes, one hole will have at least 2 pigeons in it.

b) ~~[illegible]~~ Assume that there is a (n+1)-element subset of $\{1, \ldots, 2n\}$ without two consecutive integers.

Then we can form the ~~distinct~~ sets $\{k_1, k_2, k_3, \ldots, k_{n+1}\}$ and $\{k_1+1, k_2+1, k_3+1, \ldots, k_{n+1}+1\}$ of distinct numbers (no two elements equal) both of size n+1. ~~The combined size of these two~~

Note that our range of possible numbers will be 1 to 2n+1 (because the max of k_{n+1} is 2n), a total of 2n+1 possible values.

But if the two sets are distinct, then we should have $(n+1)+(n+1) = 2n+2$ ~~distinct possible values~~ distinct numbers

~~We have a contradiction, 2n+2 values need~~

$\Rightarrow$ We have a contradiction, we need to fit 2n+2 ~~vals~~ distinct ~~values~~ numbers into 2n+1 possible ~~holes~~ values

$2n+2 > 2n+1$, so by the pigeonhole principle we will have ~~[illegible]~~ two distinct numbers with the same value. This is impossible. ↯

$\therefore$ Every (n+1)-element subset of $\{1, \ldots, 2n\}$ contains two consecutive integers. □

(c) No, the same statement is not true if we replace "(n+1)-element subset" by "n-element subset" as we could simply pick the subset of all even numbers; i.e. $\{2, 4, 6, \ldots, 2n\}$ which would be a n-element subset of $\{1, \ldots, 2n\}$ without two consecutive integers

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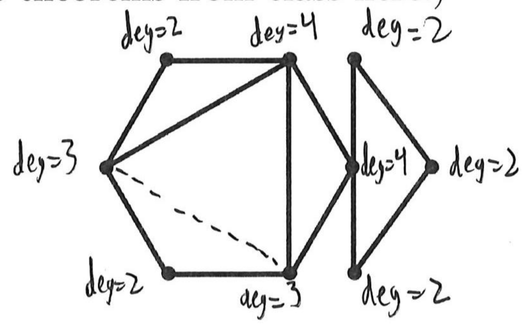
7. Let $G = (V, E)$ be a simple undirected graph.

(a) (2 points) Define *Euler cycle* and *Hamiltonian cycle*.

(b) (2 points) Show (without using theorems from class) that if G has a vertex of odd degree, then G does not have an Euler cycle.

(c) (2 points) Can you add one edge to the following graph to make it into a simple undirected graph that has an Euler cycle? Explain. (You may use theorems from class here.)



(d) (2 points) Does the graph given in part (c) have a Hamiltonian cycle? Explain.

(a) Euler cycle: An Euler cycle is a cycle that passes through every edge once.
Hamiltonian cycle: A Hamiltonian cycle is a cycle that passes through every vertex once.

(b) If G has an Euler cycle, then every time we enter a vertex via one edge we must exit via a different edge. This means that each vertex must have an even number of edges incident on it. Thus, every vertex must have even degree. ∴ If G has a vertex of odd degree, then G does not have an Euler cycle.

(c) Yes, you can add one edge to the graph to make it into a simple undirected graph that has an Euler cycle. This is because there are only two vertices of odd degree, so by placing one edge that connects these two vertices, all vertices will have even degree. If all vertices in a graph have even degree, then that graph must have an Euler cycle. ∴ We know that if we add one edge causing all vertices to have even degree, then we have made a graph with an Euler cycle.

(d) The graph given in part (c) does not have a Hamiltonian cycle because if we look at the graph, , The only way to access the vertices in the right most  is through a single vertex . So in any cycle that touches every vertex, this vertex must be passed through twice. ∴ No hamiltonian cycle exists.

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8. Let $G = (V, E)$ be an undirected connected planar graph with at least one cycle. Denote by v the number of vertices, e the number of edges, and f the number of faces of G. Assume that every cycle of G has length exactly 5.

   (a) (3 points) Explain why $2e \geq 5f$.

   (b) (2 points) State Euler's formula, and use it show that $5v - 3e \geq 2$.

   (c) (2 points) Does there exist such a graph G that is bipartite? If yes, draw an example. If not, explain why not.

(a) ~~If every cycle has length 5, then every cycle must enclose 1 face, every edge is associated w/ 2 faces. The number of faces then must equal the total number of cycles + 1 for the outer face, because every enclosed face will represent a cycle of some kind. So in the least edges, most faces scenario we have that the sum of~~

If every cycle is length 5, then each edge is incident on two faces. In the worst case scenario with the least edges and most faces each cycle is incident on 5 previously uncounted faces. We count each edge twice to account for both faces it is incident on. We get $2e \geq 5f$.

(b) ~~$5e \geq 2e - 5f + 10$~~ ~~$e = v - f + 2$~~ ~~$5e = 5v - 5f + 10$~~ ~~$5f = 5v - 5e + 10$~~ ~~$2e \geq$~~

~~$v = e - f + 2$~~
~~$5v = 5e - 5f + 10$~~
~~$5f = 5e - 5v + 10$~~
~~$2e \geq 5e - 5v + 10$~~
~~$-3e + 5v \geq 10$~~
~~$5v - 3e \geq 10$~~

Euler's formula: $V = e - f + 2$

$V = e - f + 2$
$5v = 5e - 5f + 10$
$5f = 5e - 5v + 10$
$2e \geq 5e - 5v + 10$

$5v - 3e \geq 10$
It follows that
$5v - 3e \geq 2$ □ ✓

(c)

No, there does not exist such a graph G that is bipartite because for a graph to be bipartite, every cycle must have even length, and we know every cycle of G has length exactly 5 which is odd

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9. (8 points) For each of the following statements, write TRUE or FALSE (please not just T or F). **No justification is required.**

(a) For every $n > 1$, there are exactly $2^{(n^2)}$ relations on a set of size n.

TRUE

(b) Let R be the relation on $\mathbb{Z}$ defined by xRy if and only if at least one of x and y is prime. Then R is transitive.

FALSE

(c) For every $n \geq k \geq 1$: $C(n, k) = P(n, k) \cdot k!$

FALSE

(d) There are $C(10, 5)C(5, 3)$ strings of length 10 that contain exactly 5 'a's, 3 'b's and 2 'c's.

TRUE

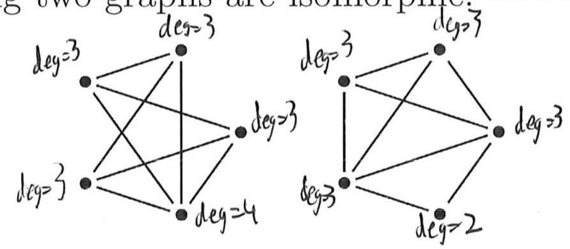
(e) For every $n \geq 1$: $\sum_{k=0}^{n} C(n, k) = 2^n$.

TRUE

(f) Let $G = (V, E)$ be a simple undirected graph. If every vertex has degree at least 2, then G contains a cycle.

TRUE

(g) The following two graphs are isomorphic:



FALSE

(h) There are exactly $\frac{C(5, 2)^3}{6}$ undirected graphs with vertex set $\{1, 2, 3, 4, 5\}$, that have 3 edges, and that have no loops.

TRUE