Final Exam

Instructions: Do not open this exam until instructed to do so. You will have 3 hours to complete the exam. Please print your name and student ID number above. You may **not** use calculators, books, notes, or any other material to help you. Please make sure your phone is silenced and stowed where you cannot see it. You may use the front and back of the page for your answers. If you need scratch paper, please ask one of the proctors. You must **show all your work** to receive credit.

Note: In this entire exam, you may leave your numerical responses written as products, fractions, and with factorials. However, you are not allowed to leave expressions of the form $C(n,k)$, $P(n,k)$, etc. As always, you need to justify all your answers.

Question	Points	Score
1	6	
2	7	
3	7	
4	7	
5	5	
6	5	
7	8	
8	7	
9	8	
Total:	60	

1. Consider the sequence a_n, $n \geq 1$, defined by the recurrence relation

$$a_n = a_{n-2} + n, \qquad \text{for } n \geq 3,$$

and with initial conditions $a_1 = 2$, $a_2 = 5$.

(a) (1 point) Is this a linear homogeneous recurrence relation with constant coefficients? If so, what is its order? If not, explain why not.

(b) (5 points) Prove by induction that for every $n \geq 1$: $a_{2n-1} = n^2 + 1$.

2. Let X, Y be sets, and $f : X \to Y$ be a function.

   (a) (1 point) Define what it means for f to be one-to-one.

   (b) (1 point) Define what it means for f to be onto.

   Let now Z also be a set and $g : X \to Z$ be a function. Define the new function

$$h : X \to Y \times Z : h(x) = (f(x), g(x)).$$

   (c) (3 points) Show that if f and g are one-to-one, then h is one-to-one.

   (d) (2 points) Prove or give a counterexample: if f and g are onto, then h is onto.

3. (a) (2 points) Let R be a relation on a set X. Define what it means for R to be reflexive, symmetric, anti-symmetric, and transitive.

Now let X be the set of 6-bit strings. Define a relation R on X by sRt if and only if s and t have the same number of zeroes. (So for instance $011100\, R\, 010101$, but $011100 \not R\, 010010$.)

(b) (3 points) Show that R is an equivalence relation.

(c) (2 points) How many elements are there in the equivalence class of 001100? Explain.

4. (a) (1 point) How many sequences of length 6 can we form with the numbers from $\{1, 2, \ldots, 9\}$? (Repetition is allowed.)

   (b) (3 points) How many such sequences start with 1 or end with 9?

   (c) (3 points) For how many of the sequences in part (a) do the 6 numbers add up to 15?

5. Consider the recurrence relation $a_n = 3a_{n-1} - 2a_{n-2}$ for $n \geq 2$ with initial conditions $a_0 = 0$ and $a_1 = 2$.

   (a) (1 point) Is this a linear homogeneous recurrence relation with constant coefficients? If so, what is its order? If not, explain why not.

   (b) (4 points) Solve the recurrence relation.

6. (a) (1 point) State (a version of) the pigeonhole principle.

   (b) (3 points) Let $n \geq 1$ be an integer. Use the pigeonhole principle to show that every $(n+1)$-element subset of $\{1, \ldots, 2n\}$ contains two consecutive integers.

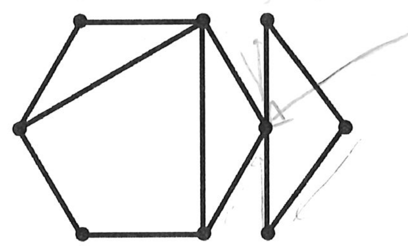
   (c) (1 point) Is the same statement still true if we replace "$(n+1)$-element subset" by "n-element subset"? Justify your answer.

7. Let $G = (V, E)$ be a simple undirected graph.

(a) (2 points) Define *Euler cycle* and *Hamiltonian cycle.*

(b) (2 points) Show (without using theorems from class) that if G has a vertex of odd degree, then G does not have an Euler cycle.

(c) (2 points) Can you add one edge to the following graph to make it into a simple undirected graph that has an Euler cycle? Explain. (You may use theorems from class here.)



(d) (2 points) Does the graph given in part (c) have a Hamiltonian cycle? Explain.

8. Let $G = (V, E)$ be an undirected connected planar graph with at least one cycle. Denote by v the number of vertices, e the number of edges, and f the number of faces of G. Assume that every cycle of G has length exactly 5.

(a) (3 points) Explain why $2e \geq 5f$.

$f - e + v = 2$

(b) (2 points) State Euler's formula, and use it show that $5v - 3e \geq 2$.

(c) (2 points) Does there exist such a graph G that is bipartite? If yes, draw an example. If not, explain why not.

9. (8 points) For each of the following statements, write TRUE or FALSE (please not just T or F). **No justification is required.**

(a) For every $n > 1$, there are exactly $2^{(n^2)}$ relations on a set of size n.

(b) Let R be the relation on $\mathbb{Z}$ defined by xRy if and only if at least one of x and y is prime. Then R is transitive.

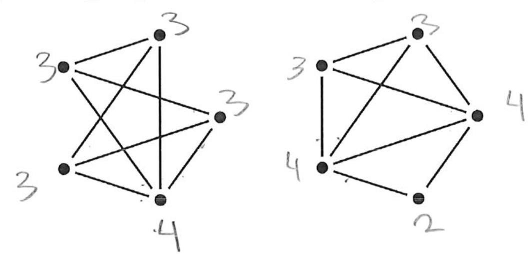
(c) For every $n \geq k \geq 1$: $C(n,k) = P(n,k) \cdot k!$

(d) There are $C(10,5)C(5,3)$ strings of length 10 that contain exactly 5 'a's, 3 'b's and 2 'c's.

(e) For every $n \geq 1$: $\sum_{k=0}^{n} C(n,k) = 2^n$.

(f) Let $G = (V, E)$ be a simple undirected graph. If every vertex has degree at least 2, then G contains a cycle.

(g) The following two graphs are isomorphic:



(h) There are exactly $\frac{C(5,2)^3}{6}$ undirected graphs with vertex set $\{1,2,3,4,5\}$, that have 3 edges, and that have no loops.