PAIK

MIDTERM 2 (MATH 61, SPRING 2015)

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Math 61 Section: 1C

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The rules:
You MUST simplify completely and BOX all answers with an **INK PEN**.
You are allowed to use only this paper and pen/pencil. No calculators.
No books, no notebooks, no web access. You MUST write your name and UCLA id.
Except for the last problem, you MUST write out your logical reasoning and/or proof in full. You have exactly 50 minutes.

Warning: those caught writing after time get automatic 10% score deduction.

Points:

44
21

1	10
2	8
3	13
4	8
5	26

Total: 65 **(out of 100)**

Problem 1. (20 points)

Compute the number of (shortest) grid walks from $(0,0)$ to $(9,9)$ which:

a) do not go through any of the other diagonal points $(1,1)$, $(2,2)$, ..., $(8,8)$

b) stay on or above $y=x-1$ diagonal

c) stay on or above $y=x$ diagonal AND do not go through $(6,6)$

d) stay on or above $y=x$ diagonal AND on or below $y=x+1$ diagonal.

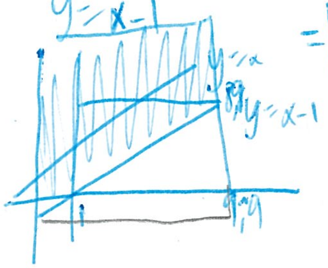
Note: You can use the formula for Catalan numbers or write the answer in terms of C_n's.

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

a) don't go through

$2 \cdot C_8$ because it never touches the diagonal and either goes above or below $y=x$, which is the same as an 8x8 grid on or above the diagonal (multiply by 2 for above and below

b) stay on or above $y=x-1$



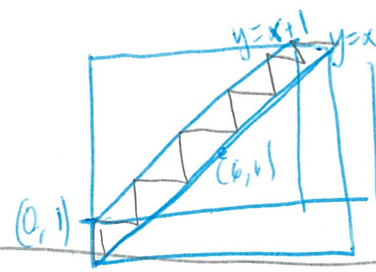
$= C_8$ because $y=x-1$ moves the diagonal to start at $(1,0)$ and end at $(9,8)$ for an 8x8 grid

$\Rightarrow \boxed{C_{10} \quad (10 \times 10 \text{ grid})}$ $\binom{2n}{n}-\binom{2n}{n+2}$

$\binom{18}{9}-\binom{18}{11}$

c) $C_9 - C_6 \cdot C_3$; total number of shortest grid walks above or on diagonal minus path that goes through $(6,6)$

d) on or above $y=x$ And on or below $y=x+1$



$C_9 - C_8$; because it can only go between the 8x8 and 9x9 grids

1, can only go up and right in one way between $y=x$ and $y=x+1$

10

Problem 2. (20 points)

Compute the number of subgraphs of G isomorphic to H, where

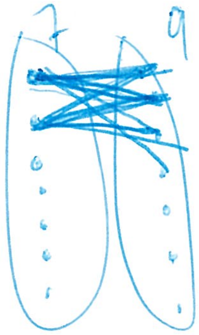
a) $G = K_{7,9}$, $H = C_4$

b) $G = K_{7,9}$, $H = P_4$

c) $G = C_9$, $H = P_4$

d) $G = K_9$, $H = K_{2,3}$

a)



$\# C_{2n}$ in $K_{nn} = \frac{n! \cdot n!}{2n}$

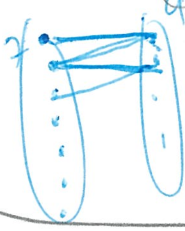
$\# C_4$ in $K_{7,9} = \boxed{\frac{\binom{7}{2}\binom{9}{2}}{2 \cdot 7}}$ b/c must choose two from left side and atleast two from the other side; ÷2 for ~~orientation~~ flipped & divide by 7

$\binom{7}{2}\binom{9}{2}$; choose two vertices on each side

says 2·7

3

b) P_4 in $K_{7,9}$



$\frac{7!\,9!}{2} \Rightarrow \boxed{\frac{(7-2)!\,(9-2)!}{2}}$ b/c no orienta

remove any one edge from C_4 to make it P_4 $\quad 4\binom{7}{2}\binom{9}{2}$

c) # of P_4 in C_9 = $\frac{n!}{2(n-r)!} = \frac{9!}{2(5)!}$

$\# P_r$ in $C_n = \frac{n!}{2(n-r)!}$

9 vertices, each can be start of 3 consecutive edges in row $\Rightarrow$ 9

$\boxed{\frac{9!}{2(5!)}}$; Path has no orientation so divide by two and since $r \le n$, must account for that by $(n-r)!$

d) # $K_{2,3}$ in K_9 = $\boxed{\binom{9}{2}\binom{7}{3}}$; choose two for left side of bipartite and 3 for right side of bipartite

9

5

13

Problem 3. (15 points)

Let $a_1 = 2$, $a_2 = 7$, $a_{n+1} = a_n + 2a_{n-1}$. Solve this LHRR and find a closed formula for a_n.

$a_{n+1} = a_n + 2a_{n-1}$

$\lambda^2 = \lambda + 2 \iff \lambda^2 - \lambda - 2 = 0$; $(\lambda + 1)(\lambda - 2) = 0$

$\lambda_{1,2} = -1, 2$

$a_n = \alpha(-1)^n + \beta(2)^n$

$a_1 = 2 = \alpha(-1)^2 + \beta(2)^2 \iff \alpha + 4\beta = 2$ $\quad \alpha + \frac{3}{44}(4)' = \frac{22}{11} - \frac{3}{11}$

$a_2 = 7 = \alpha(-1)^7 + \beta(2)^7 \iff + \ -\alpha + 128\beta = 7$ $\quad \alpha = \frac{19}{11}$

$132\beta = 9$ $\quad \frac{9}{132} = \frac{3}{44}$ $\quad 3 \,|\, 132$

$\beta = \frac{3}{44}$

$$\boxed{a_n = \cancel{\tfrac{19}{11}}\,1\,(-1)^n + \cancel{\tfrac{3}{44}}\,\tfrac{3}{2}\,(2)^n}$$

$2 = \alpha(-1)^1 + \beta(2)^1$ $\quad 2 = -\alpha + 2\beta$

$7 = \alpha(-1)^2 + \beta(2)^2$ $\quad + \ 7 = \alpha + 4\beta$

$\frac{9}{6} = \frac{6\beta}{6}$

$\beta = \frac{3}{2}$, $\alpha = 1$

$2 = -\alpha + \frac{3}{2} \cdot 2$

$2 - 3 = -1 = -\alpha$

$\alpha = 1$

$$\boxed{a_n = (-1)^n + \frac{3}{2}(2)^n}$$

Problem 4. (15 points)

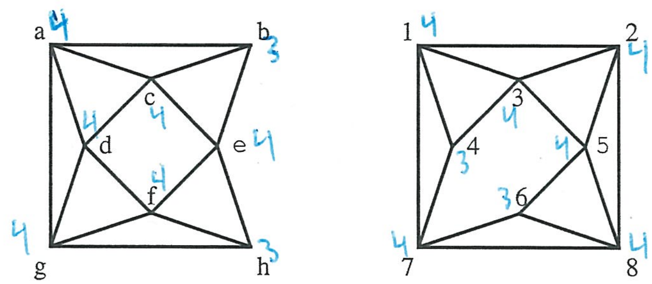
Decide whether the following pairs of graphs on 8 vertices are isomorphic or non-isomorphic.

$G_1 \simeq G_2$

They are isomorphic and form bija[ction] $\pi: V_1 \to V_2$ and $E_1 \to E_2$

a)

a→3
b→4
c→1
d→2
e→7
f→8
g→5
h→6

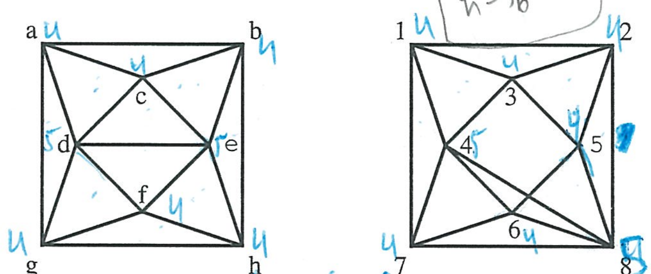


a → 1
b → 4
c → 3
d → 5
e → 2
f → 8
g → 7
h → 6

$G_1 \simeq G_2$

a→3
b→4
c→1
d→2
e→7
f→8
g→5
h→6

b)



a →
b →
c →
d →
e →
f →
g →
h →

$G_3 \not\cong G_4$

Important: In case of isomorphism, you must write a bijection in the figure above (in ink). No need for further arguments. In case non-isomorphism, you must say so and present an argument why two graphs are not isomorphic.

8

$G_3 \not\cong G_4$ because the number of subgraphs of G_3 isomorphic to C_3 ≠ the number of subgraphs of G_4 isomorphic to C_3; the vertices 4→8 messes that up while G_3 has from d→e

Problem 5. (30 points, 2 points each) **TRUE or FALSE?**

Circle correct answers with ink. No explanation is required or will be considered.

T **F** (1) Isomorphic graphs have the same number of edges.

T **F** (2) Isomorphic graphs have the same number of connected components.

T **F** (3) Isomorphic graphs have the same number of 4-cycles.

1 2 5 14 42

T **F** (4) $F_n \leq C_n$ for all integer n.

1 1 2 3

T **F** (5) Sequence $(3, 3, 3, 3, 3)$ is a valid score of a simple graph.

T **F** (6) Sequence $(4, 4, 4, 4, 2)$ is a valid score of a simple graph.

T **F** (7) Sequence $(4, 4, 4, 2, 2)$ is a valid score of a simple graph.

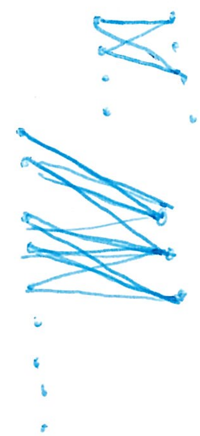
T **F** (8) Sequence $(4, 4, 2, 2, 2)$ is a valid score of a simple graph.

T **F** (9) Sequence $(2, 2, 2, 0, 0)$ is a valid score of a simple graph.

T **F** (10) Graph C_8 is a subgraph of $K_{7,7}$.

$C_{2n} \Rightarrow K_{nn}$ $8/2=4$ n $\frac{4!\,4!}{2n}$

T **F** (11) Graph C_8 is a subgraph of $K_{9,3}$.



T **F** (12) Graph P_8 is a subgraph of $K_{9,3}$.

T **F** (13) Graph K_4 is a subgraph of $K_{7,7}$.

$\binom{9}{2} = \frac{9!}{2!\,7!} = \frac{9 \cdot 8 \cdot 4}{2} = 36$

T **F** (14) Graph K_9 has 72 edges.

T **F** (15) Catalan numbers modulo 2 are periodic with period 6.

$\hookrightarrow$ $2 \times n = 0 \bmod 2$

1 2 3 4 5 6

1 2 5 14 42

$\frac{1}{n+1}\binom{2n}{n}$

$\frac{1}{8}\binom{14}{7}$ $\frac{1}{7} \cdot \frac{14!}{7!\,7!}$

C_1 C_2 C_3 C_4 C_5 C_6 C_7

mod 2 1 1 0 1 0 0 0 1

$13 \cdot 2 \cdot 11$

$143 \cdot 2 = 286$

42

42