Problem 1. (20 points)

Compute the number of (shortest) grid walks from $(0,0)$ to $(9,9)$ which:

$a)$ do not go through any of the other diagonal points $(1,1)$, $(2,2)$, $\ldots$, $(8,8)$.

Solution: If the first step of a good walk is up, the entire walk remains on or above the first super diagonal (that is, $(0,1),(1,2),\ldots,(8,9)$). Therefore the set

$$\{\text{good walks whose first step is up}\}$$

is in bijection with the set

$$\{\text{on-or-above-the-diagonal walks on an } 8 \times 8 \text{ grid}\}$$

so there are C_8 of them. Similarly, the good walks whose first step is right are in bijection with the on-or-below-the-diagonal walks on an 8×8 grid so there are C_8 good walks in this case.

Answer: $2C_8$.

$b)$ stay on or above $y = x - 1$ diagonal

Solution: The good walks are in bijection with the on-or-above-the-diagonal walks on the 10×10 grid with corners $(0,-1),(0,9),(10,9),(10,-1)$. The bijection takes a good grid walk (on or above $y = x - 1$) on the 9×9 grid and on the segment from $(0,-1)$ to $(0,0)$ and the segment from $(9,9)$ to $(10,9)$.

Answer: C_{10}.

$c)$ stay on or above $y = x$ diagonal AND do not go through $(6,6)$

Solution: The number of walks that **do** go through $(6,6)$ is C_6C_3 because they are fully determined by (independently) choosing an on-or-above-the-diagonal walk from $(0,0)$ to $(6,6)$ and an on-or-above-the-diagonal walk from $(6,6)$ to $(9,9)$. Because walks that **do not** go through $(6,6)$ are the complement of those that do, we simply subtract C_6C_3 from the total number of on-or-above-the-diagonal walks on the 9×9 grid.

Answer: $C_9 - C_6C_3$.

$d)$ stay on or above $y = x$ diagonal AND on or below $y = x + 1$ diagonal.

Solution: To stay on or above $y = x$, the first step must be up. Then to stay on or below $y = x + 1$, the second step must right. Following this pattern, each odd step is up and each even step is right so there is only one such walk.

Answer: 1.

Problem 2. (20 points)

Compute the number of subgraphs of G isomorphic to H, where

a) $G = K_{7,9}$, $H = C_4$.

Solution: Because each edge in G goes between the two sides, any graph isomorphic to H must have 2 vertices from each side, i.e. $\binom{7}{2}\binom{9}{2}$ choices of vertices. After deleting all other vertices (and their edges), each of the vertices in our subgraph has degree 2 and thus it is a C_4 so we have no more decisions to make.

Answer: $\binom{7}{2}\binom{9}{2}$.

b) $G = K_{7,9}$, $H = P_4$.

Solution: Again we must choose 2 vertices from each side, for which we again have $\binom{7}{2}\binom{9}{2}$. However, P_4 has only 3 edges so we must pick which of the 4 edges in our subgraph to delete.

Answer: $4\binom{7}{2}\binom{9}{2} = 9 * 8 * 7 * 6$

c) $G = C_9$, $H = P_4$.

Solution: A P_4 is chosen by picking 3 edges in a row in G. Because G has 9 edges each with an edge on either side this leaves 9 choices for the middle edge of H, fully determining H.

Answer: 9.

d) $G = K_9$, $H = K_{2,3}$

Solution: First we must choose 2 of the 9 vertices in G to be on "Side A" of H, i.e. $\binom{9}{2}$ choices. Then we (independently) choose 3 from the remaining 7 vertices to be on "Side B" of H, i.e. $\binom{7}{3}$ choices. To make H isomorphic to $K_{2,3}$ we must delete exactly the other 4 vertices (and their edges) as well as all of the edges within Side A and all of the edges within Side B. Therefore H is fully determined

Answer: $\binom{9}{2}\binom{7}{3} = \frac{9!}{4!3!2!} = \binom{9}{2,3}$.

Problem 3. (15 points)

Let $a_1 = 2$, $a_2 = 7$, $a_{n+1} = a_n + 2a_{n-1}$. Solve this LHRR and find a closed formula for a_n.

Solution: Because $x^2 - x - 2 = (x+1)(x-2)$ has roots -1 and 2, Theorem 7.2.11 in the textbook says any solution to the LHRR must be of the form $a_n = b(-1)^n + d(2^n)$. To find b and d, we solve the system

$$\begin{aligned} 2 =& a_1 = -b + 2d \\ 7 =& a_2 = b + 4d \end{aligned}$$

to get $d = \frac{3}{2}$, $b = 1$.

Answer: $a_n = (-1)^n + \frac{3}{2}(2^n) = (-1)^n + 3(2^{n-1})$

Problem 4. (15 points)

Decide whether the following pairs of graphs on 8 vertices are isomorphic or non-isomorphic. We will call the left graph G and the right graph H in parts (a) and (b).

a) Because b, h, 4, and 6 are the only vertices with degree three, we know either $b \to 4, h \to 6$ or $b \to 6, h \to 4$. Bbecause G is horizontally symmetric, either one is fine so we'll say $b \to 4, h \to 6$. The only shared neighbor of b and h is e so e must map to the only shared neighbor of 4 and 6, i.e. 7. b and e both neighbor c so c must map to the unique shared neighbor of 4 and 7, i.e. 1. Exchanging b for h in the previous sentence shows f maps to 8. The only shared neighbor of c and f that hasn't been labelled is d so, using the same logic as before, d must map to 2. Since a neighbors b, a must map to a neighbor of 4 so 3 is the only remaining possibility. This leaves only 5 for g to map to.

Answer:

$$a \to 3$$
$$b \to 4$$
$$c \to 1$$
$$d \to 2$$
$$e \to 7$$
$$f \to 8$$
$$g \to 5$$
$$h \to 6$$

b) G and H cannot be isomorphic because H contains a subgraph isomorphic to K_4 (the subgraph has vertex set $\{4, 6, 7, 8\}$ but G does not.

Problem 5. (30 points, 2 points each) **TRUE or FALSE?**

Circle correct answers with ink. No explanation is required or will be considered.

(1) Isomorphic graphs have the same number of edges. **True.**

(2) Isomorphic graphs have the same number of connected components. **True.**

(3) Isomorphic graphs have the same number of 4-cycles. **True.**

(4) $F_n \leq C_n$ for all integer n. **True.**

Explanation: This can be proven by induction. Using the convention $C_0 = F_0 = 1$, the base cases (for indices 0 nd 1) are $1 \leq 1$ and $1 \leq 1$. The induction hypothesis will be that for all $k < n$ (here $n > 2$) $F_k \leq C_k$. Then

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-k} \geq C_{n-1} + C_{n-2} \geq F_{n-1} + F_{n-2} = F_n.$$

(5) Sequence $(3,3,3,3,3)$ is a valid score of a simple graph. **False.**

Explanation: By the Handshake Theorem the sum of the vertices' degrees must be even for any graph as opposed to 15 in this case.

(6) Sequence $(4,4,4,4,2)$ is a valid score of a simple graph. **False.**

Explanation: If a simple graph has 5 vertices, any degree 4 vertex neighbors all other vertices so the degree 2 vertex would have to neighbor all the degree 4 vertices. Since there are 4 of them it could not have degree 2.

(7) Sequence $(4,4,4,2,2)$ is a valid score of a simple graph. **False.**

Explanation: Same as problem (6).

(8) Sequence $(4,4,2,2,2)$ is a valid score of a simple graph. **True.**

Explanation: We can build such a graph by making a C_4 with vertex set $\{a,b,c,d\}$, adding a vertex e, and connecting e to vertices b and c. Like a cartoon house.

(9) Sequence $(2,2,2,0,0)$ is a valid score of a simple graph. **True.**

Explanation: There is exactly one such graph. We build it by taking a C_3 with vertex set $\{a,b,c\}$ and adding two vertices d and e who have no neighbors. Like a smiley face with a triangle for a mouth.

(10) Graph C_8 is a subgraph of $K_{7,7}$. **True.**

Explanation: If one side of $K_{7,7}$ is labelled with letters $a, \ldots, g$ and the other is labelled with numbers $1, \ldots, 7$ then we can find a C_8 by deleting vertices $e, f, g, 5, 6, 7$ (with their edges) and deleting all edges except the cycly $a, 1, b, 2, c, 3, d, 4, a$.

(11) Graph C_8 is a subgraph of $K_{9,3}$. **False.**

Explanation: There must be exactly 4 vertices from each side so the 3 vertex side cannot provide enough.

(12) Graph P_8 is a subgraph of $K_{9,3}$. **False.**

Explanation: Same as problem (11).

(13) Graph K_4 is a subgraph of $K_{7,7}$. **False.**

Explanation: Any subgraph of a bipartite graph is bipartite however $K_{7,7}$ is while K_4 is not.

(14) Graph K_9 has 72 edges. **False.**

Explanation: K_9 has $\binom{9}{2} = 36$ edges.

(15) Catalan numbers modulo 2 are periodic with period 6. **False.**

Explanation: If this were true for any $n \geq 1$, $C_n = C_{n+6} \mod 2$. However $C_3 = 5$ while $C_9 = 4862$.