PAK

MIDTERM 1 (MATH 61, SPRING 2015)

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Math 61 Section: 1C

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The rules:
You MUST simplify completely and BOX all answers with an **INK PEN**.
You are allowed to use only this paper and pen/pencil. No calculators.
No books, no notebooks, no web access. You MUST write your name and UCLA id.
Except for the last problem, you MUST write out your logical reasoning and/or proof in full. You have exactly 50 minutes.

Warning: those caught writing after time get automatic 10% score deduction.

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Points:

1	8
2	16
3	8
4	12
5	22

24
42

Total: 66 (out of 100)

1 3
2 2
3 1

1
2
3
4
5
6
7
8
9

Problem 1. (20 points)

$n=9$

Compute the number of permutations $(x_1, x_2, \ldots, x_n)$ of $\{1, 2, \ldots, 9\}$ such that:

a) $x_1 = 2$,

b) $x_1 \cdot x_2 \cdot x_3 = 6$, = 1,2,3 arranged in any way $\Rightarrow \boxed{3! \cdot 6!}$

c) $x_1 = x_2 = x_3 \bmod 7$,

d) $x_1 < x_2 < 5$.

a) $x_1 = 2$ 2 - - - - =

$\boxed{\text{if } n=9;\ 8!, \text{ or general } n: (n-1)!}$

~~b) $x_1 \cdot x_2 \cdot x_3 =$ 2 ways 1 3, 2 2, 3 1 $\cdot$ 6! $\boxed{2(n-3)! \text{ or if } n=9: 2\cdot 6!}$~~

n = # of elements

3 c) $x_1 = x_2 = x_3 \bmod 7 =$

7 1,8 2,9

$\boxed{0}$, no 3 numbers from same set are same mod 7

~~d)~~ ~~$x_1 < x_2 < 5 =$~~ ~~$x_1\ x_2\ 5$ 6! 7~~

1 4
2 3
3 2

~~$\boxed{9 \cdot 6!}$ or $9 \cdot (n-3)!$~~

$x_2 = 4$, $3 \cdot 7!$
+
$x_2 = 3$, $2 \cdot 7!$ $\Rightarrow \boxed{6 \cdot 7!}$
+
$x_2 = 2$, $1 \cdot 7!$

Problem 2. (20 points)

Let $X = \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ be the set of all integers. For each of these relations R, decide whether they are reflexive, symmetric or transitive (or neither).

a) xRy if and only if $|x| = |y|$.

b) xRy if and only if $x + 2y = 0 \bmod 3$.

c) xRy if and only if $x^2 + 2y^2 = 0 \bmod 3$.

d) xRy if and only if $x^3 + 122y^3 = 0 \bmod 3$.

a) xRy if $|x| = |y|$

refl: xRx : $|x| = |x|$ ✓

sym: $xRy \Rightarrow yRx$ ✓

transitive $|x| = |y| \Rightarrow |y| = |z| \Rightarrow |x| = |z|$ ✓ ✓

$\boxed{\text{Reflexive, transitive, and symmetric}}$

b) xRy $x + 2y = 0 \bmod 3$

ref: xRx $x + 2x = 3x = 0 \bmod 3$ ✓

Sym $x + 2y = 0 \bmod 3 \rightarrow y + 2x = 0$ ✓

$3y + 3x = 0 \bmod 3$

tr: $x + 2y = 0 \bmod 3 \Rightarrow y + 2z = 0 \bmod 3 \Rightarrow x + 2z = 0 \bmod 3$ ✓ ?

$\boxed{\text{Reflexive, transitive, and symmetric}}$

$x \neq y \bmod 3$

c) xRy iff $x^2 + 2y^2 = 0 \bmod 3$

ref: $x^2 + 2x^2 = 3x^2 = 0 \bmod 3$ ✓

sym: $x^2 + 2y^2 = 0 \bmod 3 \Rightarrow y^2 + 2x^2 = 0 \bmod 3 \Rightarrow 3x^2 + 3y^2 = 0 \bmod 3$ ✓ $x = y \bmod 3$

transitive: $x^2 + 2y^2 = 0 \bmod 3 \Rightarrow y^2 + 2z^2 = 0 \bmod 3 \Rightarrow x^2 + 2z^2 = 0 \bmod 3$ ✓

$\boxed{\text{Reflexive, transitive, and symmetric}}$

d) $x^3 + 122y^3 = 0 \bmod 3$

ref: $x^3 + 122x^3 = 0 \bmod 3 = 123x^3 = 0 \bmod 3 \rightarrow$ ✓

sym: $x^3 + 122y^3 = 0 \bmod 3 \Rightarrow y^3 + 122x^3 = 0 \bmod 3$

$123x^3 + 123y^3 = 0 \bmod 3$ ✓

tr: $x^3 + 122y^3 = 0 \bmod 3 \Rightarrow y^3 + 122z^3 = 0 \bmod 3 \Rightarrow x^3 + 122z^3 = 0 \bmod 3$

$2x^3 + 123y^3 + 244z^3 = 0 \bmod 3$ X

$\boxed{\text{reflexive and symmetric } \sim\sim\text{and transitive}\sim\sim}$

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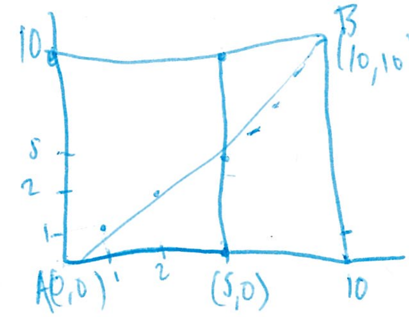

Problem 3. (15 points)

Let $A = (0,0)$, $B = (10,10)$. Find the number of (shortest) grid walks γ from A to B, such that:

a) γ never visits points $(0,10)$, $(10,1)$, $(5,5)$.
b) γ visits all points $(1,1)$, $(2,2)$, $(3,3)$, $\ldots$, $(9,9)$.
c) γ visits points $(5,0)$ and $(5,10)$, but not $(5,5)$.

$$\boxed{\binom{20}{10}-\binom{10}{0}\binom{10}{10}-\binom{11}{10}\binom{9}{0}-\binom{10}{5}\binom{10}{5}}$$

a) $(0,10)$, $(10,1)$, $(5,5)$ $\Rightarrow$ total possible − through each point

3

b) visits all points $(1,1),(2,2),(3,3),\ldots(9,9)$ = all total possible ways

$\frac{2!}{1!\,1!}=2$

$\binom{2}{1}\binom{2}{1}\cdots\binom{2}{1} \Rightarrow 10$ times $= \boxed{2^{10}}$

c) $(5,0)$ and $(5,10)$, but not $(5,5)$ $= \boxed{0}$

Visit points $(5,0)\to(5,10)$ has to go through $(5,5)$ cannot go left

$\binom{5}{0}\binom{15}{5}$

5

Problem 4. (15 points)

Recall the Fibonacci sequence: $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$, etc.

Prove that $F_n \leq 2^{n-1}$.

Proof: (by Induction)

Base case: $(n=1)$ $F_2 \leq 2^{2-1} \leq 2^0 \leq 1$

$F_1 \leq 2^{1-1} \Rightarrow 1 \leq 2^0 \Rightarrow 1 \leq 1$ ✓

need F_2

−2

Inductive Step: (n+1 step) $\Rightarrow$ assume $n \geq 2$ at this point

Want to show: $F_{n+1} \leq 2^{n+1-1} \Rightarrow F_{n+1} \leq 2^n$

$F_{n+1} = F_n + F_{n-1} \cancel{\leq 2^n}$

$= 2^{n-1} + 2^{n-2} \leq 2^n$

$= 2^n \leq 2^n \Rightarrow F_{n+1} \leq 2^n$ ∎

−1

12

Problem 5. (30 points, 2 points each) **TRUE or FALSE?**

Circle correct answers with ink. No explanation is required or will be considered.

T **F** (1) The number of functions from $\{A, B, C, D\}$ to $\{1, 2, 3\}$ is equal to 4^3.

T **F** (2) The sequence $10, 21, 32, 43, \ldots$ is increasing. increasing and non-decreasing

T **F** (3) The sequence $2/1,\ 3/2,\ 4/3,\ 5/4$ is non-increasing. 2, 1.5, 1.3, 1.2 decreasing and non-increasing

T **F** (4) There are 20 anagrams of the word $BUBUB$. $\frac{5!}{3!\,2!} = \frac{5\cdot4\cdot3\cdot2\cdot1}{3\cdot2\cdot1\cdot2\cdot1} = 10$

T **F** (5) There are more anagrams of the words $AAAACCC$ which begin with A than with C. A: $\frac{6!}{3!\,3!}$ B: $\frac{6!}{2!\,4!} = \frac{6\cdot5}{2} = 15$

T **F** (6) There are infinitely many Fibonacci numbers $= 1 \bmod 3$.

T **F** (7) There are infinitely many binomial coefficients $\binom{n}{k} = 1 \bmod 17$.

T **F** (8) Each of the 14 students wrote on a paper 10 distinct numbers, from the set $\{1, 2, \ldots, 100\}$. Then there are two students who have at least 2 numbers in common on their lists.

T **F** (9) The probability that a random 10-subset of $\{1, 2, \ldots, 19\}$ contains 10 is equal to $1/2$. $\frac{\binom{18}{9}}{\binom{19}{10}} = \frac{\binom{18}{9}}{\binom{19}{9}}$ $\frac{18!/(9!\,9!)}{19!/(9!\,10!)}$

T **F** (10) For every two subsets $A, B \subset U$, we must have $|A \setminus B| = |B \setminus A|$.

T **F** (11) For every two subsets $A, B \subset U$, we must have $|A \cup B| \geq |\overline{B}|$

T **F** (12) Every surjection that is also a bijection must be also an injection.

T **F** (13) Every surjection that is also an injection must be also a bijection.

T **F** (14) Let $\mathcal{A}$ be the set of 3-subsets of $[9] = \{1, 2, \ldots, 9\}$. Similarly, let $\mathcal{B}$ be the set of 6-subsets of $[9]$. Consider a map $f : \mathcal{A} \to [9] \setminus A$. Then f is a bijection from $\mathcal{A}$ to $\mathcal{B}$. $\binom{9}{3}$ $\binom{9}{6}$ $\frac{9!}{3!\,6!}$ $\frac{9!}{6!\,3!}$

T **F** (15) The pigeon hole principle was proved in class by induction.

$\frac{\binom{18}{9}}{\binom{19}{10}} = \frac{18!}{9!\,9!}$