Midterm 2
Discrete Mathematics (Math 61-002)

Answer the questions in the spaces provided. If you run out of room for an answer, please continue on the back of the page. **You must show your work.**

Name: ______________________ U ID: ____________

Section: ____________

Question:	1	2	3	4	Total
Points:	5	5	5	5	20
Score:	5	5	5	5	20

1. $\boxed{\text{5 points}}$ Suppose that 6 distinct integers are chosen from the set $\{1, 2, 3, \ldots, 10\}$. Show that at least two of those integers from the chosen six have a sum equal to 11.

$$11 = 1+10 = 2+9 = 3+8 = 4+7 = 5+6.$$

Let $L = \{\{1,10\},\{2,9\},\{3,8\},\{4,7\},\{5,6\}\}$.

To prevent 11 from appearing as a sum of two integers in a subset X of $\{1, 2, 3, \ldots, 10\}$, no member of L can be a subset of X. That is, we may only have a maximum of $|L| = 5$ integers in X, where each integer in X comes from a different member set of L. By the Pigeonhole principle, as soon as we try to add one more integer i to that X, $\{i, 11-i\} \subseteq X$, $\{i, 11-i\} \in L$, and as such there are always at least two integers in X with a sum of 11 in a subset of $\{1,2,3,\ldots 10\}$ with 6 members. ■

ok

2. $\boxed{\text{5 points}}$ How many **3-digit even numbers** can be formed using the digits $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$?

(**Warning:** 012 is not a 3-digit number, since $012 = 12$, but 120 is a 3-digit number and it is also an even number. You **must show your work** for this problem.)

Let i, j, k be integers.

$$\square \quad \square \quad \square$$

$$\uparrow \quad\quad \uparrow \quad\quad \uparrow$$

$$\underbrace{1 \le i \le 9}_{\boxed{9}\text{ possibilities}} \quad \underbrace{0 \le j \le 9}_{\boxed{10}\text{ possibilities}} \quad \underbrace{k \in \{0, 2, 4, 6, 8\}}_{\boxed{5}\text{ possibilities}}$$

$9 \times 10 \times 5 = \boxed{450}$ 3-digit even numbers.

3. $\boxed{\text{5 points}}$ Find the number of **integer** solutions of the following equation

$$x_1 + x_2 + x_3 = 15$$

subject to the conditions: $0 \le x_1 < 6, 1 \le x_2 < 9, x_3 \ge 0$.

$x_1 + (x_2 - 1) + x_3 = 14.$

$0 \le x_1 < 6,\ 0 \le x_2 - 1 < 8,\ x_3 \ge 0.$

Possibilities when $x_1 \ge 0,\ x_2 - 1 \ge 0,\ x_3 \ge 0.$

$$\binom{14+3-1}{3-1} = \frac{16!}{14!\,2!} = \frac{16 \cdot 15}{2} = 8 \cdot 15 = 120.$$

Possibilities when $x_1 \ge 6,\ x_2 - 1 \ge 0,\ x_3 \ge 0$

$(x_1 - 6 \ge 0)$

$$\binom{14-6+3-1}{3-1} = \binom{10}{2} = \frac{10!}{8!\,2!} = 5 \cdot 9 = 45.$$

Possibilities when $x_1 \ge 0,\ x_2 - 1 \ge 8,\ x_3 \ge 0$

$$\binom{14-8+3-1}{3-1} = \binom{8}{2} = \frac{8!}{6!\,2!} = 4 \cdot 7 = 28$$

Possibilities when $x_1 \ge 6,\ x_2 - 1 \ge 8,\ x_3 \ge 0$

$$\binom{14-8-6+3-1}{3-1} = \binom{2}{2} = 1.$$

$(x_1, x_2, x_3) = (6, 9, 0)$

By the inclusion-exclusion principle, the number of integer solutions to the given equation with the given constraints is

$$\boxed{\frac{16!}{14!\,2!} - \frac{10!}{8!\,2!} - \frac{8!}{6!\,2!} + 1}$$

$= 120 - 45 - 28 + 1 = 120 - 72 = \boxed{48}$

4. $\boxed{\text{5 points}}$ Let a_n denote the number of n-bit strings which **contain the pattern** 00. Find a **recurrence relation** that a_n satisfies. Also find the necessary **initial conditions.**

An n-bit string that contain 00.

$1\underbrace{\ldots\ldots}_{\substack{(n-1)\text{-bit string} \\ \text{that contains } 00.}}$ a_{n-1} possibilites

$01\underbrace{\ldots\ldots}_{\substack{(n-2)\text{-bit string} \\ \text{that contains } 00.}}$ a_{n-2} possibilites

$00\underbrace{\ldots\ldots}_{\text{any } (n-2)\text{-bit string}}$ 2^{n-2} possibilities

$a_n = a_{n-1} + a_{n-2} + 2^{n-2}, \quad n \geq 2$

with initial conditions:

$a_0 = 0$

$a_1 = 0$