Midterm 1
Discrete Mathematics (Math 61-002)

Name: [redacted] U ID: [redacted]

Section: [redacted]

Question:	1	2	3	4	Total
Points:	5	5	5	5	20
Score:	5	5	5	5	20

1. $\boxed{\text{5 points}}$ Prove by induction that $4^{n-1} > n^2$ for all $n \geq 3$.

<u>Base:</u> $n = 3 \Rightarrow 4^{3-1} > 3^2$

$16 > 9$ ✓

<u>Assume:</u> $n = k \Rightarrow 4^{k-1} > k^2$

<u>Induction:</u> $n = k+1 \Rightarrow 4^k \overset{?}{>} (k+1)^2$

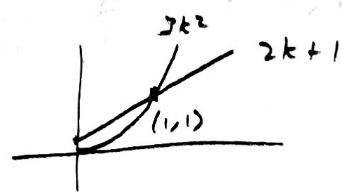
$4(4^{k-1} > k^2)$

$4^k > 4k^2$

$4^k > k^2 + 3k^2$

$(k+1)^2$

$= k^2 + 2k + 1$



for $k \geq 3$,

$3k^2 > 2k + 1$

Thus $k^2 + 3k^2 > k^2 + 2k + 1$

Since $4k > k^2 + 3k^2$, $k^2 + 3k^2 > k^2 + 2k + 1$,

$4k > k^2 + 2k + 1$

$4k > (k+1)^2$

Thus $4^{n-1} > n^2 \quad \forall n \geq 3$ by induction

2. $\boxed{\text{5 points}}$ Let $f : X \to Y$ and $g : Y \to Z$ be two functions such that $g \circ f : X \to Z$ is surjective, i.e., onto. Then prove that g is surjective.

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Since function $g \circ f$ is surjective, for every element $z \in Z$, there is $x \in X$ such that $g(f(x)) = z$.

Thus, there exists $y = f(x)$ such that $y \in Y$. Therefore, for every $z \in Z$, there exists a $y \in Y$ such that $g(y) = z$, where $y = f(x)$.

Thus the function g is surjective.

3. $\boxed{\text{5 points}}$ Let $\mathbb{N} = \{1, 2, 3, \ldots,\}$ be the set of natural numbers. Let R be a relation on $\mathbb{N}$ defined as $(x, y) \in R$ if 5 divides $x + 4y$. Prove that R is reflexive, symmetric and transitive.

$(x,x) \Rightarrow \quad x+4x = 5x,$

which is divisible by 5

$\boxed{\text{Reflexive}}$

If $(x,y) \rightarrow x+4y$ is divisible by 5,

$(y,x) \rightarrow y+4x = (5y-4y)+(5x-x)$

$= (5y+5x)-(x+4y)$

which is divisible by 5

$\boxed{\text{symmetric}}$

Excellent proof!!! Good job!

If $(x,y) \rightarrow x+4y$ and $(y,z) \rightarrow y+4z$

are divisible by 5,

$(x,z) \longrightarrow x+4z = x+4y-4y+4z$

$= (x+4y)-5y+(y+4z)$

which is divisible by 5

$\boxed{\text{transitive}}$

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$0 \quad \frac{1}{2} \quad \frac{2}{5}$

4. $\boxed{\text{5 points}}$ Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \dfrac{x}{x^2+1}$ for all $x \in \mathbb{R}$, is neither injective nor surjective.

$$\frac{x}{x^2+1} = \frac{y}{y^2+1}$$

$$y + \frac{1}{y} = x + \frac{1}{x}$$

$$\frac{1}{2} + \frac{1}{\frac{1}{2}} = 2 + \frac{1}{2}$$

$$f(2) = \frac{2}{5} = f\left(\frac{1}{2}\right)$$

Not injective

say $\quad 2 = \dfrac{x}{x^2+1}$

$$2x^2 + 2 = x$$

$$2x^2 - x + 2 = 0$$

$$x = \frac{1 \pm \sqrt{1-16}}{4}$$

$$x = \frac{1}{4} \pm \frac{\sqrt{15}}{4} i \quad \Rightarrow \quad f(x) < 2$$

Not surjective

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$$f'(x) = \frac{(x^2+1) - 2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} = 0$$

$$x = 1, -1$$

$$-\frac{1}{2} \leq f(x) \leq \frac{1}{2}$$