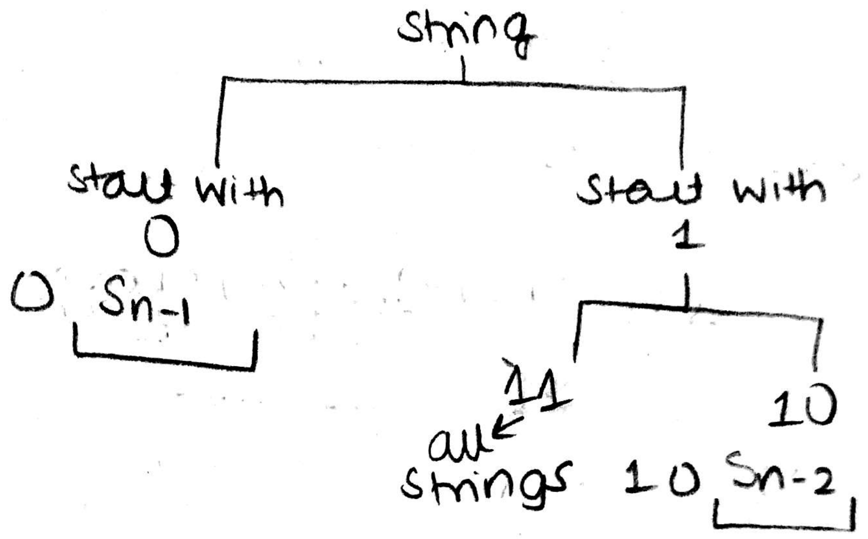
1. (10 points) Let a_n denote the number of length n strings on $\{0, 1\}$ which contain the pattern 11. Find a recurrence relation that a_n satisfies. Make sure to also give the initial conditions.

We can divide the strings as:



∴ the recurrence relation for strings containing 11 is

$$S_n = S_{n-1} + S_{n-2} + 2^{n-2}$$

S_{n-1}	S_{n-2}	2^{n-2}
Start with 0	start with 10	Start with 11

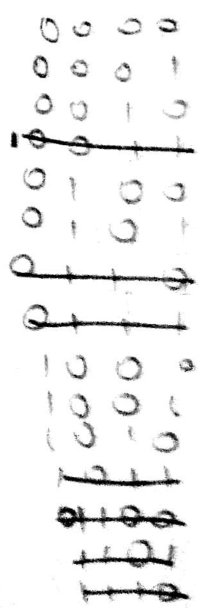
The initial conditions:

$S_1 = 0$

$S_2 = 1$

$S_3 = 3$

$S_4 = 8$



2. (5 points) Let $1 \leq k \leq m \leq n$ be integers. Give a *combinatorial* explanation[1] to show

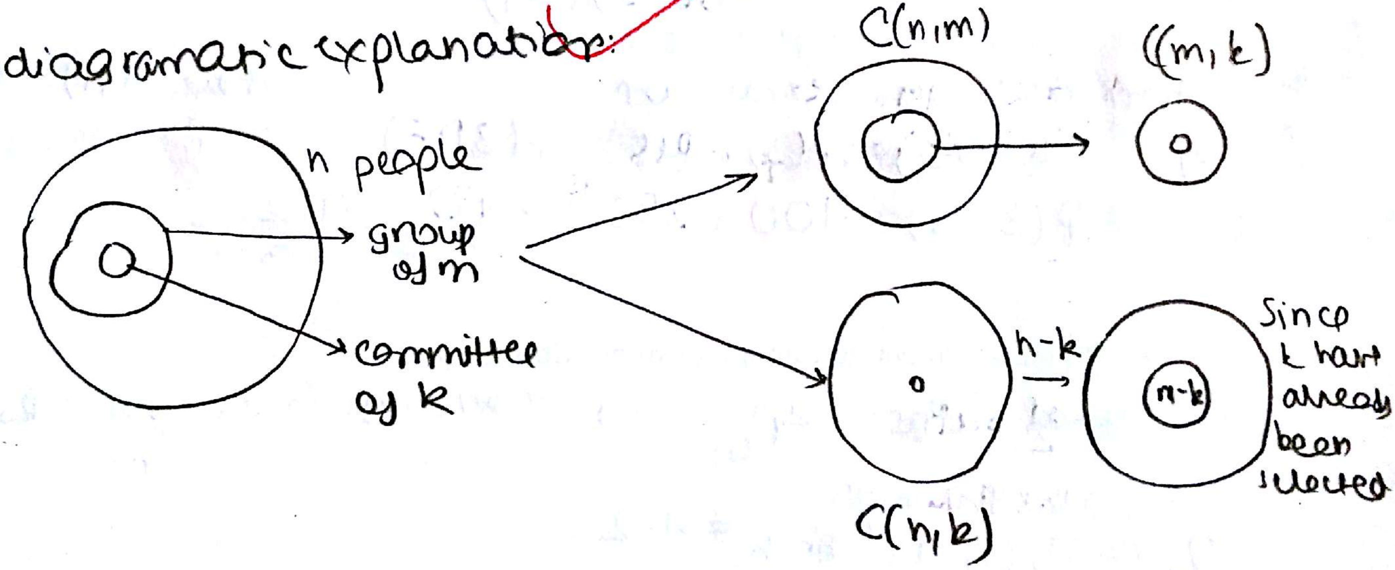
$$C(n, m)C(m, k) = C(n, k)C(n-k, m-k).$$

The left hand side can be interpreted as choosing m people from a group of n people and then making a commitee of k people from the chosen m.

C(n,m). C(m,k)

Similarly, the RHS can be interpreted as choosing the commitee of k people first i.e. C(n,k). and then choosing m-k people [along with the previously chosen k] to represent the group.

A diagramatic explanation:



[1]Please be as precise as possible, points will be taken off if the answer is not coherent/lacks details. If you use the algebraic formula for C(n, k) to show it, you will not get any points for your answer.

3. On a grid, you are allowed to move only up or right. Let $A = (0,0), B = (6,6)$. Find the number[2] of (shortest) grid paths from A to B which

(a) (2 points) go through $(3,3)$

find paths from (0,0) to (3,3) × (3,3) to (6,6)

2 $({}^{6}C_{3})({}^{6}C_{3}) = \underline{400}$

$\frac{10 \times 9 \times 8 \times 7 \times 6}{5 \times 4 \times 3 \times 2 \times 1}$ 63 ×4 252

(b) (2 points) go through $(5,5)$

~~find~~ paths from (0,0) to (5,5) ⇒ ${}^{10}C_5$

2 paths from (5,5) to (6,6) ⇒ ${}^{2}C_1$

∴ total ⇒ $({}^{10}C_5)({}^{2}C_1) = \underline{252}$

(c) (2 points) go through $(3,3)$ and $(5,5)$

paths from (0,0) to (3,3) ⇒ ${}^{6}C_3$

2 paths from (3,3) to (5,5) ⇒ ${}^{4}C_2$ → arrange UURR in different ways

paths from (5,5) to (6,6) ⇒ ${}^{2}C_1$

∴ total paths: $({}^{6}C_3)({}^{4}C_2)({}^{2}C_1) = 240$ 20 $\frac{4 \times 3}{2 \times 1}$

(d) (2 points) go through $(3,3)$ or $(5,5)$

from inclusion-exclusion: where $P(n)$ = paths thru (n,n)

2 $P(3 \cup 5) = P(3) + P(5) - P(3 \cap 5)$

⇒ $P(3 \cup 5) = 400 + 252 - 240 = \underline{412}$

(e) (2 points) do not go through $(3,3)$ nor through $(5,5)$.

total paths: ${}^{12}C_6$ → rearrange 6 Us and 6 Rs.

paths through

2 (3,3), (5,5) or both $= 412$

∴ total paths thru neighter ⇒ ${}^{12}C_6 - 412 = 924 - 412 = \underline{512}$

[2] You can leave your answer in binomial coefficients.

$\frac{12 \times 11 \times 10 \times 9 \times 8 \times 7}{6 \times 5 \times 4 \times 3 \times 2 \times 1}$ ✓

33 × 28 = 264 + 660 = 924 − 412

seven

4. (5 points) Suppose that 7 distinct integers are chosen from the set $\{1, 2, 3, ..., 12\}$. Show that at least two of the ~~six~~ numbers differ[3] by exactly 3. Also give an example to show it is possible to choose the 7 distinct integers such that no two of them differ by exactly 4.

We can make two sets such that:

$S_1 = a_1, a_2, a_3, a_4, a_5, a_6, a_7 \longrightarrow$ a_n is a number from the set $1 \leq a_n \leq 12$

and:

$S_2 = a_1+3, a_2+3, a_3+3, a_4+3, a_5+3, a_6+3, a_7+3, 4 \leq a_n \leq 15$ $+3$

Therefore, $|S_1| = 7$ and $|S_2| = 7$ [where a are in X set] ??

Since these are all in set X where $|X| = 12$, we have to have some overlap from PHP $[|S_2| + |S_1|] = 14]$.
There can't be within sets as all are distinct.

$\therefore S_i = S_j + 3$ for some i and j

X

[A better proof would include $a_1 - 3$ etc.
I don't know how to do that]

An example to not have numbers differing by 4:

$\{1, 2, 3, 4, 10, 11, 12\}$

✓

[3] Difference between n and m is $|n - m|$

5. (5 points) Solve the recurrence $4a_n = -4a_{n-1} + 3a_{n-2}$ with $a_0 = 0$ and $a_1 = -2$.

4

Let $a_n = R^n$, then we get:

$$4R^n = -4R^{n-1} + 3R^{n-2}$$

Dividing throughout by R^{n-2}:

$$4R^2 = -4R + 3$$

$$\Rightarrow 4R^2 + 4R - 3 = 0$$

We get:

$$R = \frac{-4 \pm \sqrt{4^2 - (4 \times 3 \times 4)}}{8} = \frac{-4 \pm \sqrt{16+48}}{8} = \frac{-4 \pm 8}{8} = \frac{1}{2} \text{ or } -\frac{3}{2}$$

Therefore: $R = \frac{1}{2}, -\frac{3}{2}$

We get the solution:

$$a_n = c_1\left(\frac{1}{2}\right)^n + c_2\left(\frac{-3}{2}\right)^n$$

We know:

$$a_0 = 0 \Rightarrow c_1\left(\frac{1}{2}\right)^0 + c_2\left(\frac{-3}{2}\right)^0 \Rightarrow c_1 + c_2 = 0 \quad \Rightarrow c_1 = -c_2$$

$$a_1 = -2 \Rightarrow c_1\left(\frac{1}{2}\right)^1 + c_2\left(-\frac{3}{2}\right)^1 \Rightarrow \frac{c_1}{2} - \frac{3c_2}{2} = -2 \rightarrow c_1 - 3c_2 = -4$$

$$\Rightarrow -c_2 - 3c_2 = -4$$

$$\therefore c_2 = 1$$

$$c_1 = -1$$

So we get:

$$a_n = \left(\frac{1}{2}\right)^n - \left(\frac{-3}{2}\right)^n$$

6. (10 points) Circle the correct answer. No need to show work for this question. (Also no partial credit)

8

(a) Number of length 7 strings on $\{0,1\}$ with exactly five 0s and no substring 11 is
A. C(6, 2) B. C(6, 3) C. C(7, 2) D. C(7, 3) E. None of these.

(b) Number of relations on $X = \{1, 2, 3, 4\}$ which are both reflexive and symmetric is
A. 2^{16} ~~B.~~ 2^{12} ~~C.~~ 2^{6} ~~D.~~ 2^{4} E. None of these. has to be (a) or (e)

(c) $-1 + 2C(5,1) - 4C(5,2) + 8C(5,3) - 16C(5,4) + 32$ equals
A. 1 B. −1 C. 0 D. 32 E. None of these.

(d) A simple graph has 7 vertices and 21 edges. The minimum among the degree of its vertices is
A. 6 B. 7 C. 4 D. 0 E. Cannot say.

(e) Number of ways of dividing ten identical candies among 5 children is
A. C(14, 5) B. C(14, 4) C. C(15, 4) D. C(15, 10) E. None of these.

Space for scratch work :

2^n $\quad$ 2^n $\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$

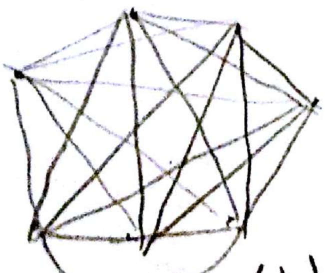
(a) ↓ 0 ↓ 0 ↓ 0 ↓ 0 ↓ 0 ↓ $\quad {}^6C_2$

(b) all reflexive: 2^{n^2-1} $\quad$ diagonal: 2^{n^2-n} $\quad 2^{n+\frac{n^2+1}{2}}$

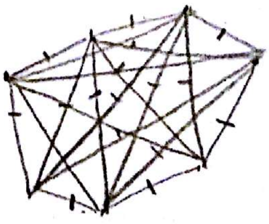
(c) $-1 + (2\cdot 5) - 4\left(\frac{5\times 4}{2}\right) + 8\left(\frac{5\times\cancel{4}\times 4}{\cancel{3}\times 2}\right) - 16\left(\frac{5\times 4\times 3\times 2}{4\times 3\times 2}\right) + 32$

$\Rightarrow -1 + 10 - 40 ~~+ 80~~ - 80 + 32 = \underline{1}$ $\qquad$ 10 ${}^{14}C_4$

$(2-1)^5 \to 1$



2^n



$2^{\frac{n^2+1}{2}}$

$\to 2^{17/2} \to 2^{8.5}$

2^{15} $2^{8.5}$

(d) $2^{15} + 2^{8.5}$