20F-MATH61-2 Midterm 1

SAMUEL ALSUP

TOTAL POINTS

12 / 13

QUESTION 1

1 Problem 1 4 / 4

✓ + **2 pts** **Base Case**

✓ + **2 pts** **Induction Step**

+ **0.5 pts** Set up base case correctly
+ **0.5 pts** Set up inductive step correctly
+ **0.5 pts** Partial credit for inductive step
+ **0.5 pts** Partial credit for basis step
- **1 pts** Missing details
- **0.5 pts** Needs justification
+ **0 pts** This question is asking about cartesian products and unions of sets, not about numbers.

QUESTION 2

2 Problem 2 4 / 4

✓ + **1 pts** **Anti-symmetry correct argument**

✓ + **1 pts** **Anti-symmetry example**

✓ + **1 pts** **Anti-reflexivity correct argument**

✓ + **1 pts** **Anti-reflexivity example**

+ **0 pts** Gives examples, but states explanation of example in incoherent way
+ **0 pts** Gives examples, but states the arguments in an incoherent/insufficient way
+ **0 pts** Incoherent explanation
+ **1 pts** Gives examples and correctly explains them, but fails to explains why such examples prove the validity of the statement.
+ **0 pts** Unmotivated explanation

QUESTION 3

3 Problem 3 2 / 3

+ **3 pts** Correct

✓ + **1 pts** **Showed f is bijective for n=1**

+ **1 pts** Showed f is surjective for n>1

✓ + **1 pts** **Showed f is not injective for n>1**

+ **0 pts** Incorrect

To justify that f is surjective, you need to show that for any number k between 1 and n, there is a subset with least element k.

QUESTION 4

4 Problem 4 2 / 2

✓ + **1 pts** **For computing the number of ways to order students with Averie first / Charlie last**

✓ + **1 pts** **For calculating the number of ways to put Averie first and Charlie last, and using the inclusion-exclusion principle**

+ **2 pts** Other valid solution (if it gives the right answer)
+ **0 pts** No solution
+ **0 pts** Other problem solved (with AND instead of OR)
+ **2 pts** Other problem solved correctly (there are two Averies and two Charlies)
+ **0 pts** Assumed that the other students are indistinguishable twins and solved the problem.
+ **1 pts** Just the right answer

Samuel Alsup
Offer
Section 2E

Math 61 Midterm 1

Problem 1: Consider an arbitrary natural number $n \geq 2$. Let $A_1, \dots, A_n$ and C be arbitrary sets. Using mathematical induction, show that $\left(\bigcup_{i=1}^{n} A_i\right) \times C = \bigcup_{i=1}^{n} (A_i \times C)$

Base case: $n = 2$

So, $\left(\bigcup_{i=1}^{2} A_i\right) \times C = \bigcup_{i=1}^{2} (A_i \times C)$

Let $(a,b) \in (A_1 \cup A_2) \times C = \bigcup_{i=1}^{2} (A_i \times C) = (A_1 \times C) \times (A_2 \times C) = \left(\bigcup_{i=1}^{2} A_i\right) \times C$

This is true $\xleftrightarrow{\text{iff}}$ $a \in (A_1 \cup A_2)$ and $b \in C$

$\longleftrightarrow$ $a \in A_1$ or $a \in A_2$ and $b \in C$, which can also be written as:

$a \in A_1$ and $b \in C$ or $a \in A_2$ and $b \in C$

So, $(a,b) \in A_1 \times C$ or $(a,b) \in A_2 \times C$, which can be written as

$(a,b) \in ((A_1 \times C) \cup (A_2 \times C))$, which as stated earlier, is equal to $\bigcup_{i=1}^{2} (A_i \times C)$

Therefore, $(a,b) \in ((A_1 \times C) \cup (A_2 \times C)) = \bigcup_{i=1}^{2} (A_i \times C)$

So, we proved $\left(\bigcup_{i=1}^{2} A_i\right) \times C \subseteq \bigcup_{i=1}^{2} (A_i \times C)$ and that

$\bigcup_{i=1}^{2} (A_i \times C) \subseteq \left(\bigcup_{i=1}^{2} A_i\right) \times C$. Therefore, by the identity $A = B$ iff $A \subseteq B$ and $B \subseteq A$,

$\left(\bigcup_{i=1}^{2} A_i\right) \times C = \bigcup_{i=1}^{2} (A_i \times C)$

Inductive step: Assume $\left(\bigcup_{i=1}^{n} A_i\right) \times C = \bigcup_{i=1}^{n} (A_i \times C)$, then

$\left(\bigcup_{i=1}^{n+1} A_i\right) \times C = \bigcup_{i=1}^{n+1} (A_i \times C)$ where $n \geq 2$.

$\left(\bigcup_{i=1}^{n+1} A_i\right) \times C = \left(\left(\bigcup_{i=1}^{n} A_i\right) \cup A_{n+1}\right) \times C = \left(\left(\left(\bigcup_{i=1}^{n} A_i\right) \times C\right) \cup (A_{n+1} \times C)\right)$

So, currently we have that $(\bigcup_{i=1}^{n+1} A_i) \times C = ((\bigcup_{i=1}^{n} A_i) \times C) \cup (A_{n+1} \times C)$

Now, using our inductive assumption, we know that:

$$\underbrace{((\bigcup_{i=1} A_i) \times C) \cup (A_{n+1} \times C) = (\bigcup_{i=1}^{n} (A_i \times C)) \cup (A_{n+1} \times C)}$$

$$= (\bigcup_{i=1}^{n+1} A_i) \times C$$

So, we have proved that: $(\bigcup_{i=1}^{n+1} A_i) \times C = \bigcup_{i=1}^{n+1} (A_i \times C)$

which was our objective in this inductive step.

Therefore, since I have proved the base case and inductive step, I have proved by mathematical induction that for an arbitrary natural number $n \geq 2$ and arbitrary sets $A_1, \ldots, A_n$, and C:

$$(\bigcup_{i=1}^{n} A_i) \times C = \bigcup_{i=1}^{n} (A_i \times C)$$

Problem 2: Let R be a relation on a set X.

a) Explain in words why the statement "R is antisymmetric" is not the negation of the statement "R is symmetric". Provide examples to illustrate your explanation.

"R is antisymmetric" is not the negation of the statement "R is symmetric" because for a relation R to be antisymmetric: if $(x,y) \in R$ and $(y,x) \in R$, then $x=y$. This description is not the negation of the statement for a relation R to be symmetric, which is if $(x,y) \in R$, $(y,x) \in R$. So, the negation of being symmetric is for some $(x,y) \in R$, $y(x) \notin R$. This is clearly different than antisymmetric definition.

For example: a set $X = \{2,4,6\}$, relation R on $X \times X = \{(2,4), (4,2), (2,6)\}$

This relation is not symmetric because for $(2,6) \in R$, $(6,2) \notin R$. It's not antisymmetric, because for $(2,4) \in R$, $(4,2) \in R$, $2 \neq 4$. The relation R is the negation of symmetric, but is not anti-symmetric, so this proves that "R is antisymmetric" is not the negation of the statement "R is symmetric".

Additionally, I will show an example where the relation R is both symmetric and antisymmetric

$R = \{(2,2), (4,4)\}$ is a relation R where for all $(x,y) \in R$, $(y,x) \in R$ and for all $(x,y) \in R$ and $(y,x) \in R$, $x=y$.

b) Explain in words why the statement "R is anti-reflexive" is not the negation of the statement "R is reflexive". Provide examples to illustrate your explanation.

1 Problem 1 **4 / 4**

- ✓ **+ 2 pts** **Base Case**
- ✓ **+ 2 pts** **Induction Step**
- **+ 0.5 pts** Set up base case correctly
- **+ 0.5 pts** Set up inductive step correctly
- **+ 0.5 pts** Partial credit for inductive step
- **+ 0.5 pts** Partial credit for basis step
- **- 1 pts** Missing details
- **- 0.5 pts** Needs justification
- **+ 0 pts** This question is asking about cartesian products and unions of sets, not about numbers.

So, currently we have that $\left(\bigcup_{i=1}^{n+1} A_i\right) \times C = \left(\left(\bigcup_{i=1}^{n} A_i\right) \times C\right) \cup (A_{n+1} \times C)$

Now, using our inductive assumption, we know that:

$$\underbrace{\left(\left(\bigcup_{i=1} A_i\right) \times C\right) \cup (A_{n+1} \times C) = \left(\bigcup_{i=1}^{n} (A_i \times C)\right) \cup (A_{n+1} \times C)}$$

$$= \left(\bigcup_{i=1}^{n+1} A_i\right) \times C$$

So, we have proved that: $\left(\bigcup_{i=1}^{n+1} A_i\right) \times C = \bigcup_{i=1}^{n+1} (A_i \times C)$

which was our objective in this inductive step.
Therefore, since I have proved the base case and inductive step, I have proved by mathematical induction that for an arbitrary natural number $n \geq 2$ and arbitrary sets $A_1, \ldots, A_n$, and C:

$$\left(\bigcup_{i=1}^{n} A_i\right) \times C = \bigcup_{i=1}^{n} (A_i \times C)$$

Problem 2: Let R be a relation on a set X.

a) Explain in words why the statement "R is antisymmetric" is not the negation of the statement "R is symmetric". Provide examples to illustrate your explanation.

"R is antisymmetric" is not the negation of the statement "R is symmetric" because for a relation R to be antisymmetric: if $(x,y) \in R$ and $(y,x) \in R$, then $x = y$. This description is not the negation of the statement for a relation R to be symmetric, which is if $(x,y) \in R$, $(y,x) \in R$. So, the negation of being symmetric is for some $(x,y) \in R$, $(y,x) \notin R$. This is clearly different than antisymmetric definition.
For example: a set $X = \{2,4,6\}$, relation R on $X \times X = \{(2,4),(4,2),(2,6)\}$
This relation is not symmetric because for $(2,6) \in R$, $(6,2) \notin R$. It's not antisymmetric, because for $(2,4) \in R$, $(4,2) \in R$, $2 \neq 4$. The relation R is the negation of symmetric, but is not anti-symmetric, so this proves that "R is antisymmetric" is not the negation of the statement "R is symmetric".
Additionally, I will show an example where the relation R is both symmetric and antisymmetric:
$R = \{(2,2),(4,4)\}$ is a relation R where for all $(x,y) \in R$, $(y,x) \in R$ and for all $(x,y) \in R$ and $(y,x) \in R$, $x = y$.

b) Explain in words why the statement "R is anti-reflexive" is not the negation of the statement "R is reflexive". Provide examples to illustrate your explanation.

"R is anti-reflexive" is not the negation of the statement "R is reflexive" because when a relation is reflexive, for all $x \in X$, $(x,x) \in R$, but when a relation is anti-reflexive, for all $x \in X$, $(x,x) \notin R$. So, we can come up with examples where a relation R on a set X is neither reflexive nor anti-reflexive. This proves that "anti-reflexive" and "reflexive" aren't negations of each other:

Example 1: a set $X = \{2,4\}$, relation R on $X \times X = \{(2,2),(2,4)\}$
This relation R is not reflexive because for $4 \in X$, $(4,4) \notin R$, but relation R is not anti-reflexive either because for $2 \in X$, $(2,2) \in R$. So, since this example demonstrates a relation that is neither reflexive nor anti-reflexive, we can be sure that the statement "R is anti-reflexive" is not the negation of the statement "R is reflexive".

Problem 3: Let n be a positive natural number. Let $X = \{i \in \mathbb{N} : 1 \leq i \leq n\}$. Denote by $P(X)$ the power set of X, and let $P^*(X) := P(X) \setminus \{0\}$ denote the set whose elements are subsets of X that are not empty. Consider the function
$f : P^*(X) \to X$
which sends each non-empty to its least element. For instance, $f(\{1,3\}) = 1$. For which values of n is f injective, surjective, or bijective? Carefully motivate your arguments.

To figure out how this works, I'll use $n=2$ as an example. If $n=2$ then $X = \{1,2\}$. This means the power set of X $(P(X)) = \{\emptyset, \{1\} \{2\}, \{1,2\}\}$.
This means $P^*(x) = \{\{1\}, \{2\} \{1,2\}\}$. The function would give us 2 for $\{2\}$ and 1 for $\{1\}$ and $\{1,2\}$. So, this function f is not injective for $n=2$ since $\{1\}$ and $\{1,2\}$ both map to the same output. We can infer that this will also be the case for all values of n greater than 2, since for 3, $\{1\}$, $\{1,2\}$, and $\{1,2,3\}$ all would have output 1. But, for $n=1$, only $\{1\}$ will have output 1, so $n=1$ is the only case where the function f is injective. Now, we must look at when f is surjective. A function is surjective if for every $x \in X$ there is at least one $y \in Y$ such that $f(x)=y$, which essentially means that every element in the codomain must have at least one element in the domain mapped to it. So, for with all of our examples with $n=1,2,3$ we have seen that every output (value in the codomain) has at least one unique set that maps to it. For example, when $n=2$, the codomain value 1 has the preimages $\{1\}$ and $\{1,2\}$ mapped to it. So, clearly every value on the codomain will have a unique preimage for any $n \geq 1$, as even for $n=1$, codomain value 1 will have preimage $\{1\}$ mapped to it. So, for all values $n \in \mathbb{N}$, f
is surjective.

2 Problem 2 **4 / 4**

✓ **+ 1 pts Anti-symmetry correct argument**

✓ **+ 1 pts Anti-symmetry example**

✓ **+ 1 pts Anti-reflexivity correct argument**

✓ **+ 1 pts Anti-reflexivity example**

+ 0 pts Gives examples, but states explanation of example in incoherent way

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+ 1 pts Gives examples and correctly explains them, but fails to explains why such examples prove the validity of the statement.

+ 0 pts Unmotivated explanation

"R is anti-reflexive" is not the negation of the statement "R is reflexive" because when a relation is reflexive, for all $x \in X$, $(x,x) \in R$, but when a relation is anti-reflexive, for all $x \in X$, $(x,x) \notin R$. So, we can come up with examples where a relation R on a set X is neither reflexive nor anti-reflexive. This proves that "anti-reflexive" and "reflexive" aren't negations of each other;

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This relation R is not reflexive because for $4 \in X$, $(4,4) \notin R$, but relation R is not anti-reflexive either because for $2 \in X$, $(2,2) \in R$. So, since this example demonstrates a relation that is neither reflexive nor anti-reflexive, we can be sure that the statement "R is anti-reflexive" is not the negation of the statement "R is reflexive".

Problem 3: Let n be a positive natural number, Let $X = \{i \in \mathbb{N} : 1 \leq i \leq n\}$. Denote by $P(X)$ the power set of X, and let $P^*(X) := P(X) \setminus \{\emptyset\}$ denote the set whose elements are subsets of X that are not empty. Consider the function
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This means $P^*(X) = \{\{1\}, \{2\} \{1,2\}\}$. The function would give us 2 for $\{2\}$ and 1 for $\{1\}$ and $\{1,2\}$. So, this function f is not injective for $n=2$ since $\{1\}$ and $\{1,2\}$ both map to the same output. We can infer that this will also be the case for all values of n greater than 2, since for 3, $\{1\}$, $\{1,2\}$, and $\{1,2,3\}$ all would have output 1. But, for $n=1$, only $\{1\}$ will have output 1, so $n=1$ is the only case where the function f is injective. Now, we must look at when f is surjective. A function is surjective if for every $x \in X$ there is at least one $y \in Y$ such that $f(x) = y$, which essentially means that every element in the codomain must have at least one element in the domain mapped to it. So, for with all of our examples with $n = 1,2,3$ we have seen that every output (value in the codomain) has at least one unique set that maps to it. For example, when $n=2$, the codomain value 1 has the preimages $\{1\}$ and $\{1,2\}$ mapped to it. So, clearly every value on the codomain will have a unique preimage for any $n \geq 1$, as even for $n=1$, codomain value 1 will have preimage $\{1\}$ mapped to it. So, for all values $n \in \mathbb{N}$, f

Now, for a function to be bijective, it must be both injective and surjective. Thus far, we have shown that the function f is injective for $n=1$ only, and that f is surjective for all $n \in \mathbb{N}$. Since it must be injective and surjective to be bijective, it is clear that f is only bijective when $n=1$.

So, the function f is injective when $n=1$, surjective for all $n \in \mathbb{N}$, and bijective when $n=1$.

Problem 4: A teacher wants to arrange their 11 students in a single line. There are two students called Averie and Charlie in this class. How many ways are there for the students to line up so that Averie is first in line or Charlie is last?

First, we need to calculate how many ways that Averie can be first in line, then how many times Charlie can be last in line, then how many times both are in these spots. This follows the principle $|A \cup B| = |A| + |B| - |A \cap B|$, where A is the set where Averie is first in line and B is the set where Charlie is last in line.

So, $|A|$ or the amount of ways Averie can be first in line is

$$\underset{1}{\text{Averie}} \cdot \underset{2}{10} \cdot \underset{3}{9} \cdot \underset{4}{8} \cdot \underset{5}{7} \cdot \underset{6}{6} \cdot \underset{7}{5} \cdot \underset{8}{4} \cdot \underset{9}{3} \cdot \underset{10}{2} \cdot \underset{11}{1} = 10! \text{ ways}$$

$|B|$ or the amount of ways Charlie can be last in line is

$$\underset{1}{10} \cdot \underset{2}{9} \cdot \underset{3}{8} \cdot \underset{4}{7} \cdot \underset{5}{6} \cdot \underset{6}{5} \cdot \underset{7}{4} \cdot \underset{8}{3} \cdot \underset{9}{2} \cdot \underset{10}{1} \quad \underset{11}{\text{Charlie}} = 10! \text{ ways}$$

$|A \cap B|$ or the amount of ways that Averie is first in line and Charlie is last in line is:

$$\underset{1}{\text{Averie}} \ \underset{2}{9} \cdot \underset{3}{8} \cdot \underset{4}{7} \cdot \underset{5}{6} \cdot \underset{6}{5} \cdot \underset{7}{4} \cdot \underset{8}{3} \cdot \underset{9}{2} \cdot \underset{10}{1} \quad \underset{11}{\text{Charlie}} = 9! \text{ ways}$$

So, $|A \cup B|$ or the amount of ways that Averie can be first in line or Charlie can be last in line is $10! + 10! - 9! = 2 \cdot 10! - 9! = 6{,}894{,}720$

So, final answer:

$\boxed{10! + 10! - 9! \text{ ways or } 2 \cdot 10! - 9! \text{ ways or } 6{,}894{,}720 \text{ ways}}$

3 Problem 3 **2 / 3**

+ **3 pts** Correct

✓ + **1 pts** **Showed f is bijective for n=1**

+ **1 pts** Showed f is surjective for n>1

✓ + **1 pts** **Showed f is not injective for n>1**

+ **0 pts** Incorrect

To justify that f is surjective, you need to show that for any number k between 1 and n, there is a subset with least element k.

Now, for a function to be bijective, It must be both injective and surjective. Thus far, we have shown that the function f is injective for $n=1$ only, and that f is surjective for all $n \in \mathbb{N}$. Since it must be injective and surjective to be bijective, it is clear that f is only bijective when $n=1$.

So, the function f is injective when $n=1$, surjective for all $n \in \mathbb{N}$, and bijective when $n=1$.

Problem 4: A teacher wants to arrange their 11 students in a single line. There are two students called Averie and Charlie in this class. How many ways are there for the students to line up so that Averie is first in line or Charlie is last?

First, we need to calculate how many ways that Averie can be first in line, then how many times Charlie can be last in line, then how many times both are in these spots. This follows the principle $|A \cup B| = |A| + |B| - |A \cap B|$, where A is the set where Averie is first in line and B is the set where Charlie is last in line.

So, $|A|$ or the amount of ways Averie can be first in line is

$$\underset{1}{\text{Averie}} \cdot \underset{2}{10} \cdot \underset{3}{9} \cdot \underset{4}{8} \cdot \underset{5}{7} \cdot \underset{6}{6} \cdot \underset{7}{5} \cdot \underset{8}{4} \cdot \underset{9}{3} \cdot \underset{10}{2} \cdot \underset{11}{1} = 10! \text{ ways}$$

$|B|$ or the amount of ways Charlie can be last in line is

$$\underset{1}{10} \cdot \underset{2}{9} \cdot \underset{3}{8} \cdot \underset{4}{7} \cdot \underset{5}{6} \cdot \underset{6}{5} \cdot \underset{7}{4} \cdot \underset{8}{3} \cdot \underset{9}{2} \cdot \underset{10}{1} \quad \underset{11}{\text{Charlie}} = 10! \text{ ways}$$

$|A \cap B|$ or the amount of ways that Averie is first in line and Charlie is last in line is:

$$\underset{1}{\text{Averie}} \; \underset{2}{9} \cdot \underset{3}{8} \cdot \underset{4}{7} \cdot \underset{5}{6} \cdot \underset{6}{5} \cdot \underset{7}{4} \cdot \underset{8}{3} \cdot \underset{9}{2} \cdot \underset{10}{1} \quad \underset{11}{\text{Charlie}} = 9! \text{ ways}$$

So, $|A \cup B|$ or the amount of ways that Averie can be first in line or Charlie can be last in line is $10! + 10! - 9! = 2 \cdot 10! - 9! = 6,894,720$

So, final answer:

$\boxed{10! + 10! - 9! \text{ ways or } 2 \cdot 10! - 9! \text{ ways or } 6,894,720 \text{ ways}}$

4 Problem 4 **2 / 2**

- ✓ **+ 1 pts** **For computing the number of ways to order students with Averie first / Charlie last**
- ✓ **+ 1 pts** **For calculating the number of ways to put Averie first and Charlie last, and using the inclusion-exclusion principle**
  - **+ 2 pts** Other valid solution (if it gives the right answer)
  - **+ 0 pts** No solution
  - **+ 0 pts** Other problem solved (with AND instead of OR)
  - **+ 2 pts** Other problem solved correctly (there are two Averies and two Charlies)
  - **+ 0 pts** Assumed that the other students are indistinguishable twins and solved the problem.
  - **+ 1 pts** Just the right answer