Instructions:

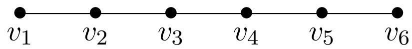
- You have from Thursday 17 December 2020 at 00.00am to 11.59pm Pacific Time to solve this exam.
- Scan your solutions and upload them to Gradescope by Thursday 17 December at 11.59pm Pacific Time. You should submit readable scans, and not pictures of your solutions. Please make sure to match the problems on the exam template with the respective parts in your solutions.
- This exam is open book, and you are allowed to use the textbook, and all resources from the lecture, or similar resources.
- You are not allowed to ask for help from others, nor give help to others taking this exam. Students suspected of academic dishonesty may be reported to the Dean of Students. This leads to a process which could end in suspension or dismissal.

Code of honor

Academic integrity is of the uttermost importance. By taking part in this evaluation, you are accepting the following code of honor:

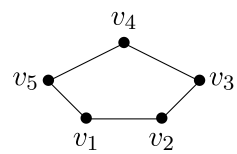
I certify on my honor that I have neither given nor received any help, or used any non-permitted resources, while completing this evaluation.

Problem 1. (5 points) Let T be the following tree:



Consider all the possible rooted trees (T, r) that you can form by choosing one of the vertices $v_1, \ldots, v_6$ as root r. Which of these trees are isomorphic as rooted trees? You should motivate your answer.

Problem 2. (5 points) Let $n \geq 3$. Consider a cycle graph, namely a connected graph with n vertices $v_1, \ldots, v_n$, connected through n edges so that they all lie in a cycle. For instance, for $n = 5$ we would have the following graph:



Explain why if we give such a graph, together with the total order $v_1 < \cdots < v_n$, as input to the breadth-first-search algorithm and depth-first-search algorithm, then the algorithms will give different spanning trees as output.

Problem 3. (5 points) Define the following relation R on $\mathbb{Z}$:

$$aRb \iff a = b + i \cdot 3 \text{ for some } i \in \mathbb{N}.$$

Show that R is a partial order.

Problem 4. (10 points) Let $n \in \mathbb{N}$ be a positive natural number, and let

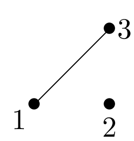
$$X_n = \{i \in \mathbb{N} \mid 1 \leq i \leq n\}.$$

Let S_n be the set whose elements are all graphs that have X_n as set of vertices. Define the function

$$f : S_n \to X_n$$

that sends a graph to its number of connected components.

For instance, if $n = 3$, then $X_3 = \{1, 2, 3\}$. Then, if we consider for example the following graph G:



we have that $f(G) = 2$.

For which values of n is f surjective, injective, or bijective? You should motivate your answer.

Problem 5. (5 points) Let n be a natural number greater or equal to 2. Assume that $A_1, \ldots, A_n$ are sets such that $A_{i_1} \cap \cdots \cap A_{i_k} \neq \emptyset$ for all $1 \leq i_1 < \cdots < i_k \leq n$ and all $1 \leq k < n$; in other words, all intersections of fewer than n sets are non-empty.

Either show that

$$A_1 \cap \cdots \cap A_n \neq \emptyset\,,$$

or give a counterexample. In either case you should motivate your answer.

Problem 6. (7 points)
A drawer contains 6 pink, 7 green, 10 blue, 12 red and 15 black pens. What is the minimum number of pens that you have to choose from the drawer to make sure that, no matter which pens you choose, you get at least 8 pens of same color?

Problem 7. (8 points)
It's New Year's morning, and you go to the corner backery to get croissants for your ten friends staying at your place. Your friend Emerson wants a chocolate croissant, your friend Austin a cream croissant and a plain croissant, while your friend Drew doesn't want to eat anything, and the remaining friends would like to have a croissant each, but are happy with any type of croissant. Considering that the backery has four different types of croissants (chocolate, cream, plain and jam), and we consider croissants of the same type to be indistinguishable, in how many different ways can you choose *at most* 15 croissants so that each friend has the croissants of their choice?

Problem 8. (5 points) Show by induction that $n! > 2^n$ for all $n \geq 4$.