Math 61 Midterm 1
OCT 21 2015

50 minutes

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UCLA ID: [redacted]

SECTION: Cross one box below

Day \ T.A.	John	Zach	Sam
Tuesday	1A	1C	1E
Thursday	1B	1D	1F

Rules: You MUST simplify completely and BOX all answers with an **INK PEN**. You are allowed to use only this paper and pen/pencil. A one-sided hand-written formula sheet is allowed. No calculators, no books, no notebooks, no web access. You MUST write your name and UCLA id. Except for the last problem, you MUST write out your logical reasoning and/or proof in full. You have exactly 50 minutes.

Warning: At 1:50pm your OUTATIME, those caught writing after time get automatic 10% score deduction.

Problem	**Value**	**Score**
Problem 1	8	8
Problem 2	10	10
Problem 3	10	10
Problem 4	12	12
Problem 5	10	10
Total	50	50

Problem 1.

(a) Show that

$$1 \cdot 2^1 + 2 \cdot 2^2 + 3 \cdot 2^3 + \cdots + n \cdot 2^n = (n-1) \cdot 2^{n+1} + 2.$$

This is $\sum_{k=1}^{n} k 2^k = (n-1) 2^{n+1} + 2$

Base case $n=1$: $1 \cdot 2^1 = 0 \cdot 2^2 + 2 = \boxed{2}$

Inductive step:

Assume $\sum_{k=1}^{n} k 2^k = (n-1) 2^{n+1} + 2$

4/4

So $\sum_{k=1}^{n+1} k 2^k = (n-1) 2^{n+1} + 2 + (n+1) 2^{n+1} = ((n+1)+(n-1)) 2^{n+1} + 2$

$= (2n) 2^{n+1} + 2 = \boxed{n \cdot 2^{n+2} + 2}$

So by induction, the above is true.

(b) Show by induction that $e^n \geq n + 1$ for integers $n \geq 1$ $(e = 2.71\ldots)$.

Base case: $n = 1$ $e^1 = 2.71.. > 2$

Inductive case:

Assume $e^n \geq n+1$, $e^{n+1} = e^n \cdot e \geq (n+1)e = en + e$

4/4

$\geq n+2$

So $\boxed{e^{n+1} \geq (n+1)+1}$

Thus by induction, the given is $\boxed{\text{true}}$

Since

10

Problem 2. Find the number of grid paths from $(0,0)$ to $(6,6)$ that

(a) go through $(2,2)$,

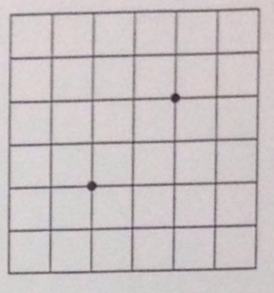
(b) go through $(4,4)$,

(c) go through $(2,2)$ or $(4,4)$,

(d) do not go through $(2,2)$ and $(4,4)$.

You can write your answers in terms of binomials.

(6,6)



(0,0)

(a) Paths to (2,2) $= \binom{4}{2}$

Paths from (2,2) to (6,6) $= \binom{8}{4}$

total $= \boxed{\binom{4}{2}\binom{8}{4}}$

(b) Paths to (4,4) $= \binom{8}{4}$

Paths from (4,4) to (6,6) $= \binom{4}{2}$

total $= \boxed{\binom{8}{4}\binom{4}{2}}$

(c) We can add the values from (a) and (b) to get a number of paths that go through (2,2) or (4,4) however, we have counted twice the paths that go through both, so we must subtract $|(a) \cap (b)|$

Thus,

$$2\cdot\binom{8}{4}\binom{4}{2} - \underbrace{\overbrace{\binom{4}{2}\binom{4}{2}\binom{4}{2}}^{\text{\# of paths throug both}}}_{\text{\# of ways } 0 \text{ to } 2,2 \,\cdot\, 2,2\text{-}4,4 \,\cdot\, 4,4\text{-}6,6} = \boxed{2\cdot\binom{8}{4}\binom{4}{2} - \binom{4}{2}^3}$$

(d) (c) counts # of paths through 2,2 or 4,4, so # of paths that go through neither $= |U - c| = \binom{12}{6} - \left[2\cdot\binom{8}{4}\binom{4}{2} - \binom{4}{2}^3\right]$

Problem 3. Compute the number of 4-subsets of $\{1, 2, 3, \ldots, 10\}$ that:

(a) contain 5,

(b) contain only one prime number 2, 3, 5, 7,

(c) the minimum or maximum is 5,

(d) the product of three of the entries is 6.

You can write your answers in terms of binomials.

a) we know $5 \in$ subset so we chose 3 from the remaining 9 elements so $\boxed{\binom{9}{3}}$

b) we pick one prime + 3 composites, ~~so~~ we get

$\boxed{\binom{4}{1} \cdot \binom{6}{3}}$

10

c) if min $= 5$, the other 3 elements > 5, so we chose 3 from the remaining 5 elements

$= \binom{5}{3}$

but if max $= 5$ we must choose 3 from the 4 remaining elements less than 5, so $\binom{4}{3}$

total $= \boxed{\binom{5}{3} + \binom{4}{3}}$

d) The only 3 elements whose product is 6 is $\{1, 2, 3\}$

so for the last element, we chose 1 from 7

$\binom{7}{1} = \boxed{7}$

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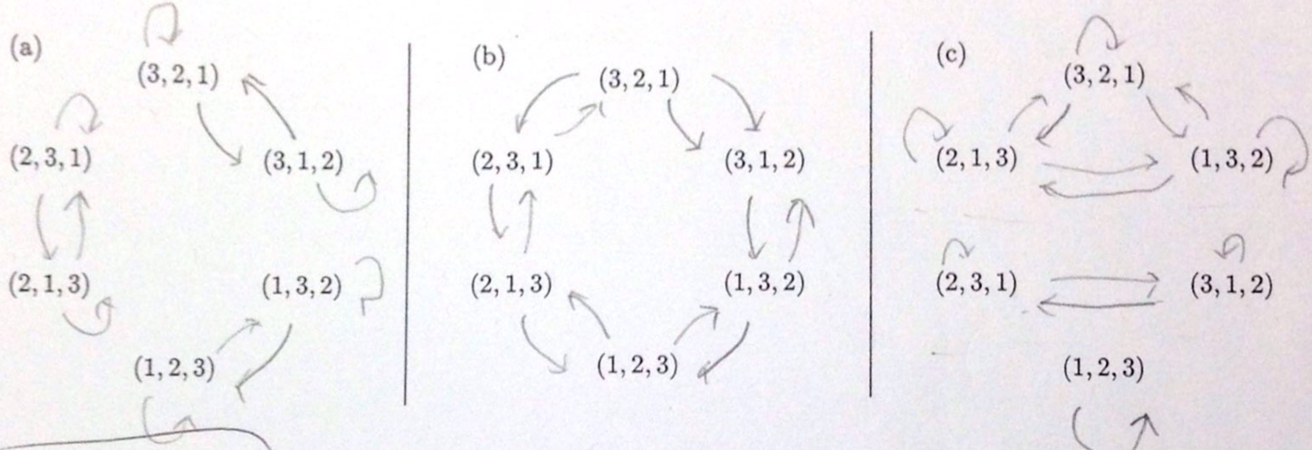
Problem 4. Let $X = \{(1,2,3), (2,1,3), (1,3,2), (2,3,1), (3,1,2), (3,2,1)\}$ be the set of permutations of size 3. For each of these relations R on X, draw its digraph and decide whether each is reflexive, symmetric or transitive (or neither).

(a) $(a_1, a_2, a_3) R (b_1, b_2, b_3)$ if and only if $a_1 = b_1$.

(b) pRq if and only if q can be obtained from p by swapping two adjacent elements. e.g. $(\mathbf{1}, \mathbf{2}, 3)\, R\, (\mathbf{2}, \mathbf{1}, 3)$.

(c) The relation induced from the partition $X_1 = \{(1,2,3)\}$, $X_2 = \{(2,1,3), (3,2,1), (1,3,2)\}$, $X_3 = \{(2,3,1), (3,1,2)\}$ of X.



(a)
reflexive ✓
symmetric ✓
transitive ✓

✓

(b)
Symmetric ✓

✓

(c)
reflexive ✓
symmetric ✓
transitive ✓

✓

12/12

10

Problem 5. True or False Circle the answers only **with ink**, next to the questions. No reasoning/calculations will be taken into account.

(a) The sequence $a_n = n! - 2^n$ is decreasing. **T. or F.**

(b) The sequence $1/\binom{2}{2}, 1/\binom{3}{2}, 1/\binom{4}{2}, \ldots$ is nonincreasing. **T. or F.** T

(c) Given sets $A, B, C \subset U$ the set $A \cup \overline{B \cup C}$ equals the set $(A \cap \overline{B}) \cup (A \cap \overline{C})$. **T. or F.**

(d) The name EMMETT has more than 88 rearrangements of its letters. **T. or F.**

(e) A prime number p divides all the binomial numbers $\binom{p}{1}, \binom{p}{2}, \ldots, \binom{p}{p-1}$. **T. or F.**

(f) There are more injections than surjections from $\{A, B, C, D\}$ to $\{1, 2, 3, 4\}$. **T. or F.**

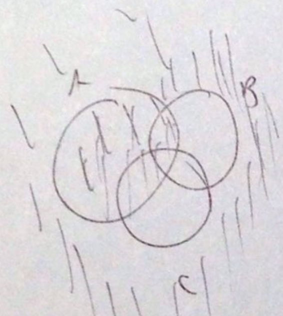
(g) There are more subsets of $\{1, 2, \ldots, 11\}$ of odd size than even size. **T. or F.**

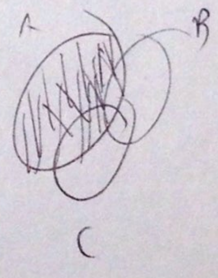
(h) There are the same # nonnegative integer solutions to $x_1 + x_2 + x_3 = 4$ as positive ingeter solutions to $y_1 + y_2 + y_3 = 7$. **T. or F.** T

(i) The coefficient of x^2y^2 in $(x + y + 1)^6$ is $\binom{6}{4}$. **T. or F.**

(j) There are more symmetric relations than antisymmetric relations on n elements. **T. or F.** F

$(x+y+1)$ ⋮ $(x+y+1)$ $(x+y+1)$ $\}6$





$\frac{6!}{8} = 6.5.9$

$\binom{6}{2}\binom{4}{4}$

$15 \cdot 6$

$\frac{p!}{(p-n)!\,n!}$

1 2 3 4 | |

1 2 3 4 5 6 7 | |

$\frac{(p-1)!}{(p-n)!\,n!}$

$\binom{6}{2}$

$p \mid 2^p - 2 - p$

$p \mid 2(2^{p-1} - 1)$