Math 61 Midterm 1
OCT 21 2015

50 minutes

Your Name: Michael Wang

UCLA ID: ______________________

SECTION: Cross one box below

Day \ T.A.	John	Zach	Sam
Tuesday	1A	1C	1E
Thursday	1B	1D	1F

Rules: You MUST simplify completely and BOX all answers with an **INK PEN**. You are allowed to use only this paper and pen/pencil. A one-sided hand-written formula sheet is allowed. No calculators, no books, no notebooks, no web access. You MUST write your name and UCLA id. Except for the last problem, you MUST write out your logical reasoning and/or proof in full. You have exactly 50 minutes.

Warning: At 1:50pm your OUTATIME, those caught writing after time get automatic 10% score deduction.

Problem	Value	Score
Problem 1	8	8
Problem 2	10	10
Problem 3	10	10
Problem 4	12	9
Problem 5	10	10
Total	50	47

Problem 1.

$\sum_{k=1}^{n} k2^k$

(a) Show that
$$1 \cdot 2^1 + 2 \cdot 2^2 + 3 \cdot 2^3 + \cdots + n \cdot 2^n = (n-1) \cdot 2^{n+1} + 2. \quad (*)$$

Base case: for $n=1$, $1 \cdot 2^1 = 0 \cdot 2^1 + 2 \Rightarrow 2 = 2$, so $(*)$ holds for $n=1$

Induction step: Assume $(*)$ is true for $n=k$,
then we show that $(*)$ is true for $n=k+1$

Since $1\cdot 2^1 + 2 \cdot 2^2 + \ldots + k2^k = (k-1)2^{k+1} + 2 \quad (*)$

add $(n+1)2^{n+1}$ to both sides, $\underbrace{1\cdot 2^1 + 2\cdot 2^2 + \ldots + k2^k + (k+1)2^{k+1}}_{LHS} = \underbrace{(k-1)2^{k+1} + (k+1)2^{k+1} + 2}_{RHS}$

$$\text{RHS}: \quad (k-1)2^{k+1} + (k+1)2^{k+1} + 2 = 2^{k+1}(2k) + 2$$
$$= k2^{k+2} + 2$$

4/4

showing that $1 \cdot 2^1 + 2 \cdot 2^2 + \ldots k2^k + (k+1)2^{k+1} = k2^{k+2} + 2$

By induction, $(*)$ is true for all $n \geq 1$.

(b) Show by induction that $e^n \geq n+1$ for integers $n \geq 1$ $\quad (e = 2.71\ldots)$.

Base case: for $n=1$, $e^1 > 1+1 \Rightarrow 2.71 > 2$ so $e^n \geq n+1$ is true for $n=1$

Induction step: Assume $e^n \geq n+1$ is true for $n=k$,
then show that $e^n \geq n+1$ is true for $n=k+1$ (i.e., $e^{k+1} \geq (k+1)+1$)

Since $e^k \geq k+1$, $\quad e^k \cdot e \geq e(k+1)$
$$\Rightarrow e^{k+1} \geq ek + e$$

Now $ek + e \geq k+2$ because $ek \geq k$ and $e \geq 2$

4/4

So $e^{k+1} \geq ek + e \geq k+2$
$$\Rightarrow e^{k+1} \geq k+2$$

By induction, $e^n \geq n+1$ is true for all $n \geq 1$.

10

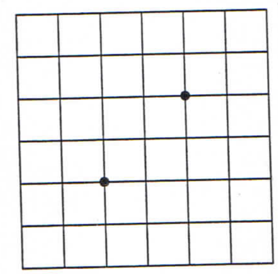
Problem 2. Find the number of grid paths from $(0,0)$ to $(6,6)$ that

(a) go through $(2,2)$,

(b) go through $(4,4)$,

(c) go through $(2,2)$ or $(4,4)$,

(d) do not go through $(2,2)$ and $(4,4)$.



You can write your answers in terms of binomials.

a) 4 blocks, 2 r ; 8 blocks, 4 r

$\binom{4}{2}\binom{8}{4}$

$\boxed{\binom{4}{2}\binom{8}{4}}$

b) Same (symmetric) $\binom{4}{2}\binom{8}{4}$

$\boxed{\binom{4}{2}\binom{8}{4}}$

c) $A \cap B = \binom{4}{2}\cdot\binom{4}{2}\cdot\binom{4}{2}$

$|A \cup B| = |A| + |B| - |A \cap B|$

$|A| = |B| = \binom{4}{2}\binom{8}{4}$ $\quad 2\cdot\binom{4}{2}\binom{8}{4} - \binom{4}{2}\binom{4}{2}\binom{4}{2}$

$\boxed{2\cdot\binom{4}{2}\binom{8}{4} - \binom{4}{2}\binom{4}{2}\binom{4}{2}}$

d) 12 blocks, 6 r $\quad D = \text{Total} - C$

$D = \binom{12}{6} - \left(2\cdot\binom{4}{2}\binom{8}{4} - \binom{4}{2}\binom{4}{2}\binom{4}{2}\right)$

$\boxed{\binom{12}{6} - \left(2\cdot\binom{4}{2}\binom{8}{4} - \binom{4}{2}\binom{4}{2}\binom{4}{2}\right)}$

Problem 3. Compute the number of 4-subsets of $\{1, 2, 3, \ldots, 10\}$ that:

(a) contain 5,

(b) contain only one prime number 2, 3, 5, 7,

(c) the minimum or maximum is 5,

(d) the product of three of the entries is 6.

You can write your answers in terms of binomials.

a) 3 subsets of 9 elem

$\binom{9}{3}$ $\boxed{\binom{9}{3}}$

b) one prime

three nonprime

$\binom{4}{1}\binom{6}{3}$ $\boxed{\binom{4}{1}\binom{6}{3}}$

c) Min. = $\binom{5}{3}$ +

Max. = $\binom{4}{3}$

$\boxed{\binom{5}{3} + \binom{4}{3}}$

d) $\{1, 2, 3, x\}$

$\binom{7}{1}$ $\boxed{\binom{7}{1}}$

16

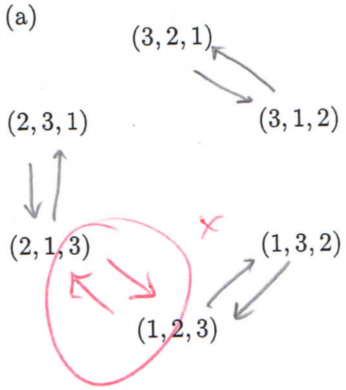
Problem 4. Let $X = \{(1,2,3),(2,1,3),(1,3,2),(2,3,1),(3,1,2),(3,2,1)\}$ be the set of permutations of size 3. For each of these relations R on X, draw its digraph and decide whether each is reflexive, symmetric or transitive (or neither).

(a) $(a_1,a_2,a_3)R(b_1,b_2,b_3)$ if and only if $a_1 = b_1$.

(b) pRq if and only if q can be obtained from p by swapping two adjacent elements. e.g. $(\mathbf{1},\mathbf{2},3)\,R\,(\mathbf{2},\mathbf{1},3)$.

(c) The relation induced from the partition $X_1 = \{(1,2,3)\}, X_2 = \{(2,1,3),(3,2,1),(1,3,2)\}$, $X_3 = \{(2,3,1),(3,1,2)\}$ of X.

(a)



-1

symmetric	y
transitive	y
reflexive	y

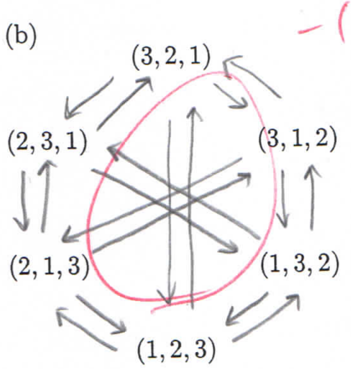
reflexive

symmetric

transitive

(b)

-1



y

n

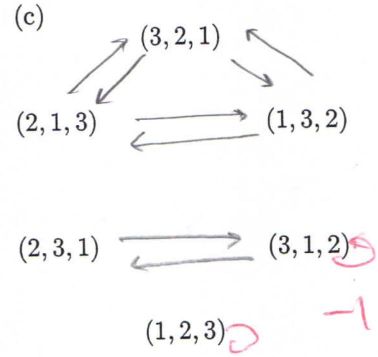
n

symmetric

(not reflexive,

not transitive) ✓

(c)



-1

y

y

y

reflexive

symmetric ✓

transitive

9/12

10

Problem 5. True or False Circle the answers only **with ink**, next to the questions. No reasoning/calculations will be taken into account.

$6-8 \quad 2^n - 16$

(a) The sequence $a_n = n! - 2^n$ is decreasing. $-1, -2, -2,$ **T. or F.**

(b) The sequence $1/\binom{2}{2}, 1/\binom{3}{2}, 1/\binom{4}{2}, \ldots$ is nonincreasing. $\frac{1}{1}, \frac{1}{3}, \frac{1}{6}$ **T. or F.**

(c) Given sets $A, B, C \subset U$ the set $A \cup \overline{B \cup C}$ equals the set $(A \cap \overline{B}) \cup (A \cap \overline{C})$. **T. or F.**

$\cancel{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$ (d) The name EMMETT has more than 88 rearrangements of its letters. $\frac{6!}{2!2!2!}$ **T. or F.**

(e) A prime number p divides all the binomial numbers $\binom{p}{1}, \binom{p}{2}, \ldots, \binom{p}{p-1}$. **T. or F.**

Inj< (f) There are more injections than surjections from $\{A, B, C, D\}$ to $\{1, 2, 3, 4\}$. **T. or F.**

(g) There are more subsets of $\{1, 2, \ldots, 11\}$ of odd size than even size. **T. or F.**

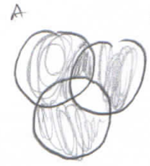
(h) There are the same nonnegative integer solutions to $x_1 + x_2 + x_3 = 4$ as positive ingeter solutions to $y_1 + y_2 + y_3 = 7$. **T. or F.**

(i) The coefficient of x^2y^2 in $(x + y + 1)^6$ is $\binom{6}{4}$. $x \cdot x \cdot y \cdot y \cdot 1 \cdot 1 \quad \frac{6!}{2!2!2!}$ **T. or F.**

(j) There are more symmetric relations than antisymmetric relations on n elements. **T. or F.**

$A + (\overline{B + C}) = A + \bar{B} \cdot \bar{C} = (A + \bar{B}) \cdot$

$A \cup (\bar{B} \cap \bar{C}) = (A \cup \bar{B}) \cap$



g) $\binom{11}{1} + \binom{11}{3}$

1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35

j) $2^k, \ k = n + \binom{n}{2}$ sym

$= n + \frac{n(n-1)}{2} = \frac{2n + n^2 - n}{2} = \frac{n^2 + n}{2}$

$\{1, 2\}$

Sym: $(1,1) \ (2,2)$
$(1,2) \cdot (2,1)$

$\{\}$
$\{1\} \ \{2\}$
$\{1,2\}$
$\{$

~~$2^k, \ k = n^2 - n$~~

$= n(n-1)$

$k = n + 2 \cdot \binom{n}{2}$ for anti