Math 61 Midterm 1

OCT 21 2015

50 minutes

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SECTION: Cross one box below

Day \ T.A.	John	Zach	Sam
Tuesday	1A	1C	1E
Thursday	~~1B~~	1D	1F

forgot section

Rules: You MUST simplify completely and BOX all answers with an **INK PEN**. You are allowed to use only this paper and pen/pencil. A one-sided hand-written formula sheet is allowed. No calculators, no books, no notebooks, no web access. You MUST write your name and UCLA id. Except for the last problem, you MUST write out your logical reasoning and/or proof in full. You have exactly 50 minutes.

Warning: At 1:50pm your OUTATIME, those caught writing after time get automatic 10% score deduction.

Problem	Value	Score
Problem 1	8	4
Problem 2	10	10
Problem 3	10	10
Problem 4	12	12
Problem 5	10	8
Total	50	44

Problem 1.

(a) Show that

$$1 \cdot 2^1 + 2 \cdot 2^2 + 3 \cdot 2^3 + \cdots + n \cdot 2^n = (n-1) \cdot 2^{n+1} + 2.$$

$$\sum_{k=1}^{n} k(2)^n$$

$$2 + \sum_{k=2}^{n+1} (k-1) 2^{n+1} = (k+2)^n$$

It employs the binomial theorem and changing index comes up to RHS when doing summation

1

(b) Show by induction that $e^n \geq n+1$ for integers $n \geq 1$ $\quad (e = 2.71\ldots)$.

Base case.

$e^1 \geq 1+1 \qquad 2.71\ldots \geq 2$ ✓

1

Induction Step:

Assume $e^n \geq n+1$ is true

Show $e^{n+1} \geq n+2$ is true

$$e^{n+1} = e^n e \geq (n+1) + 1$$

$$e^n e \geq e^n + 1$$

$$e \geq 1 + \frac{1}{e^n} \quad \leq 1$$

1 + a number less than 1 will always be less than 2, which is less than 2.71…

Hence by proof of induction, $e^n \geq n+1$ for all ints $n \geq 1$

4/4

10

Problem 2. Find the number of grid paths from (0, 0) to (6, 6) that

(a) go through (2, 2),

(b) go through (4, 4),

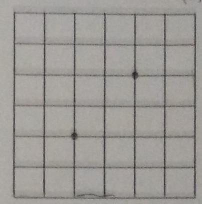
(c) go through (2, 2) or (4, 4),

(d) do not go through (2, 2) and (4, 4).

You can write your answers in terms of binomials.

Think: picking where the "ups" will be

(6, 6)



(0, 0)

a) $\boxed{\binom{4}{2}\cdot\binom{8}{4}} = \frac{4\times 3}{1\times 2}\cdot$

b) $= \boxed{\binom{8}{4}\cdot\binom{4}{2}} =$

$|X\cup Y| = |X| + |Y| - |X\cap Y|$

c) $= \boxed{\underbrace{\binom{4}{2}\cdot\binom{8}{4}}_{\text{thrugh }(2,2)} + \underbrace{\binom{4}{2}\cdot\binom{8}{4}}_{\text{thrugh }(4,4)} - \underbrace{\binom{4}{2}\cdot\binom{4}{2}\cdot\binom{4}{2}}_{\text{thrugh both }(2,2)\text{ and }(4,4)}}$

d) = do not go thrugh ~~either~~ both = total − goes thrugh both

total = $\binom{12}{6}$ = "Choose 6 spots out of 12 for the ups"

$\boxed{\underbrace{\binom{12}{6}}_{\text{total}} = \underbrace{\left[\binom{4}{2}\cdot\binom{8}{4}+\binom{4}{2}\cdot\binom{8}{4}-\binom{4}{2}\cdot\binom{4}{2}\binom{4}{2}\right]}_{\text{goes thrugh either aka part d.}}} = \underbrace{\text{does not go thrugh either}}$ (based on test day)

Problem 3. Compute the number of 4-subsets of $\{1, 2, 3, \ldots, 10\}$ that:

(a) contain 5,

(b) contain only one prime number 2, 3, 5, 7,

(c) the minimum or maximum is 5,

(d) the product of three of the entries is 6.

You can write your answers in terms of binomials.

a) contains 5, 3 to choose from 9, $\boxed{\binom{9}{3}}$ =

b) contains only 1 prime number 2,3,5,7 Non prime: 1,4,6,8,9,10

Pick the 1 prime number Pick the remaining non primes

$\boxed{\binom{4}{1} \cdot \binom{6}{3}}$

c) 4 subsets w 5 as min + 4 subsets with 5 as max

6, 7, 8, 9, 10 ← pick 3 1, 2, 3, 4 ← pick 3

$\boxed{\binom{5}{3} + \binom{4}{3}}$

HC

d) Product of 3 entries is 6 only works with $1 \cdot 2 \cdot 3$

Thus we have to have those in the set.

and $\binom{7}{1}$ from the remaining

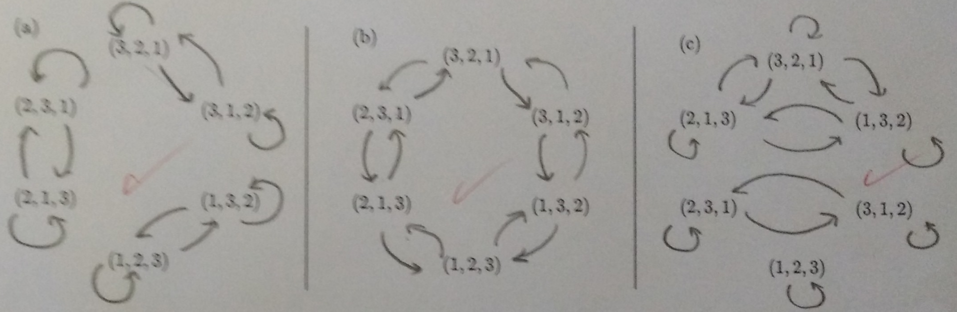
$\boxed{\binom{7}{1} = 7 \text{ 4' sets}}$

Problem 4. Let $X = \{(1,2,3),(2,1,3),(1,3,2),(2,3,1),(3,1,2),(3,2,1)\}$ be the set of permutations of size 3. For each of these relations R on X, draw its digraph and decide whether each is reflexive, symmetric or transitive (or neither).

(a) $(a_1,a_2,a_3)R(b_1,b_2,b_3)$ if and only if $a_1 = b_1$.

(b) pRq if and only if q can be obtained from p by swapping two adjacent elements. e.g. $(1,2,3)\,R\,(2,1,3)$.

(c) The relation induced from the partition $X_1 = \{(1,2,3)\}$, $X_2 = \{(2,1,3),(3,2,1),(1,3,2)\}$, $X_3 = \{(2,3,1),(3,1,2)\}$ of X.



a) is.... reflexive, symmetric, and transitive (by default)

b) is.... NOT reflexive, symmetric, NOT transitive!

No items swapped!

c) is..... an equivalence relation, so it is

Reflexive, symmetric, and transitive!

12/12

8

Problem 5. True or False Circle the answers only **with ink**, next to the questions. No reasoning/calculations will be taken into account.

(a) The sequence $a_n = n! - 2^n$ is decreasing. T. or $\boxed{\text{F.}}$

(b) The sequence $1/\binom{2}{2}, 1/\binom{3}{2}, 1/\binom{4}{2}, \ldots$ is nonincreasing. $\boxed{\text{T.}}$ or F.

(c) Given sets $A, B, C \subset U$ the set $A \cup \overline{B \cup C}$ equals the set $(A \cap \overline{B}) \cup (A \cap \overline{C})$. $\boxed{\text{T.}}$ or F.

(d) The name EMMETT has more than 88 rearrangements of its letters. $\boxed{\text{T.}}$ or F.

(e) A prime number p divides all the binomial numbers $\binom{p}{1}, \binom{p}{2}, \ldots, \binom{p}{p-1}$. ? $\boxed{\text{T.}}$ or F.

(f) There are more injections than surjections from $\{A, B, C, D\}$ to $\{1, 2, 3, 4\}$. T. or $\boxed{\text{F.}}$

(g) There are more subsets of $\{1, 2, \ldots, 11\}$ of odd size than even size. T. or $\boxed{\text{F.}}$

(h) There are the same nonnegative integer solutions to $x_1 + x_2 + x_3 = 4$ as positive ingeter solutions to $y_1 + y_2 + y_3 = 7$. $\boxed{\text{T.}}$ or F.

(i) The coefficient of x^2y^2 in $(x + y + 1)^6$ is $\binom{6}{4}$. T. or $\boxed{\text{F.}}$

(j) There are more symmetric relations than antisymmetric relations on n elements. $\boxed{\text{T.}}$ or F.

$\frac{6!}{\ }$ $\qquad \frac{6 \times 5}{1 \times 2} = 15$

$\frac{6!}{2!2!} = =$ $\qquad 6 \qquad 2! - 2^0$

$5 \times 4 \times 3 \times 2 \times 1$

$\frac{6!}{2!2!2!} = \frac{6!}{6} = 5! = 120 \qquad 2^5 = 32$

$6 - 8$

$120 - 32$

$\frac{1}{1} \quad \frac{1}{3} \quad \frac{1}{6} \quad \frac{4 \times 3}{1 \times 2}$

$A \cup (\overline{B} \cap \overline{C})$

$= (A \cup \overline{B}) \cup (A \cap \overline{C})$