Math 61 Midterm 1
OCT 21 2015

50 minutes

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SECTION: Cross one box below

Day \ T.A.	John	Zach	Sam
Tuesday	1A	1C	1E
Thursday	1B	1D	1F

Rules: You MUST simplify completely and BOX all answers with an **INK PEN**. You are allowed to use only this paper and pen/pencil. A one-sided hand-written formula sheet is allowed. No calculators, no books, no notebooks, no web access. You MUST write your name and UCLA id. Except for the last problem, you MUST write out your logical reasoning and/or proof in full. You have exactly 50 minutes.

Warning: At 1:50pm your OUTATIME, those caught writing after time get automatic 10% score deduction.

Problem	**Value**	**Score**
Problem 1	8	8
Problem 2	10	3
Problem 3	10	6
Problem 4	12	11
Problem 5	10	7
Total	50	35

Problem 1.

(a) Show that

$$1 \cdot 2^1 + 2 \cdot 2^2 + 3 \cdot 2^3 + \cdots + n \cdot 2^n = (n-1) \cdot 2^{n+1} + 2.$$

base case: $n=1$, $(1-1)\,2^2+2 = 2$

Given n, show $n+1$

$F(n)+t_{n+1} = (n-1)\cdot 2^{n+1} + 2 + (n+1)\,2^{n+1} = 2^{n+1}(n\cancel{-1}+n\cancel{+1}) + 2$

$= 2 \cdot 2^{n+1} \cdot n + 2 = n2^{n+2} + 2 = F(n+1)$

By induction,

$1\cdot 2^1 + 2\cdot 2^2 + 3\cdot 2^3 + \ldots + n\cdot 2^n = (n-1)\cdot 2^{n+1} + 2$

(b) Show by induction that $e^n \geq n+1$ for integers $n \geq 1$ $\quad (e = 2.71\ldots)$.

(T)

Base case: $n=1$ $\quad e^n \geq 2$ ✓

Inductive step: Given n, show $n+1$

$e^{n+1} = e^n \cdot e$ $\qquad$ because $e^n \geq n+1$

we try to prove

(4) ~~Since~~

$e^n \cdot e \geq n+2$ $\qquad e^{n+1} \geq n+2$ ✓

$e^n \geq \frac{n+2}{e}$ which is true since and $e^n \geq n+1$

$e^n \geq \frac{n+2}{e}$

if $e^n \geq n+1$, it must follow that $e^{n+1} \geq n+2$

$\boxed{\frac{n+2}{e} < n+1}$ ✓

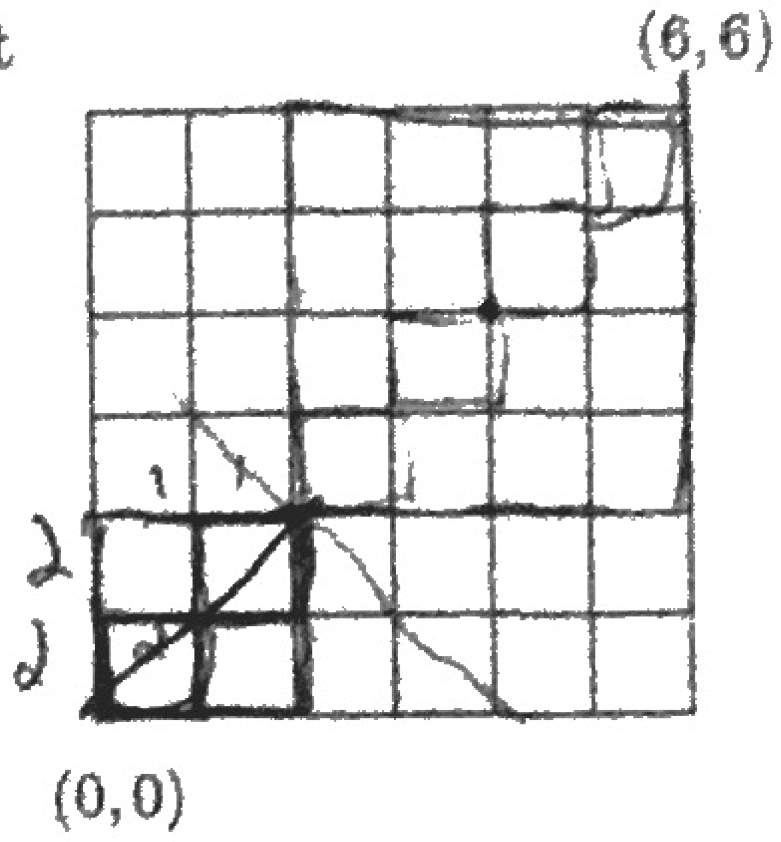
be $e^n \geq \frac{n+2}{e}$ therefore by induction $\boxed{e^n \geq n+1 \text{ for } n \geq 1}$

$n+2 < en + e$

$\boxed{0 < (e-1)n + (e-2)}$ ✓

3

Problem 2. Find the number of grid paths from (0,0) to (6,6) that



(a) go through (2,2),

(b) go through (4,4),

(c) go through (2,2) or (4,4),

(d) do not go through (2,2) ~~and~~ ?or (4,4).

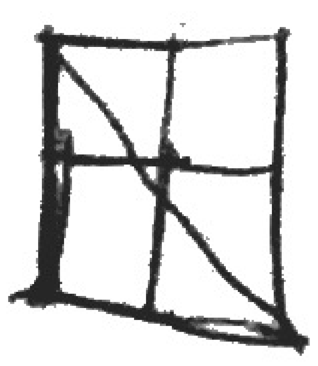
You can write your answers in terms of ~~binomials~~.

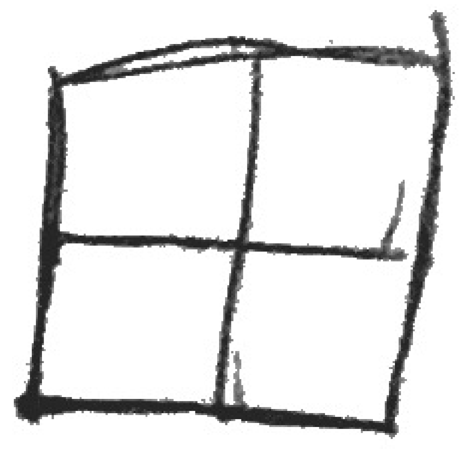
a) $\binom{4}{2}+\binom{8}{6}$ a 2^4+2^8

$2\cdot 2\cdot 2$

-1

b) $\binom{4}{2}+\binom{8}{6}$ b





-1

$2\cdot 2(1+2+1)$ $\frac{4!}{2!2!}$

2^4

d) $\binom{12}{8}$

-2

$|X|+|Y|-|X\cap Y|$

$2^4+2^8+2^4+2^8-(2^4+2^4+2^4$

c $2\left(\binom{4}{2}+\binom{8}{6}\right)$

$-\left(3\binom{4}{2}\right)$

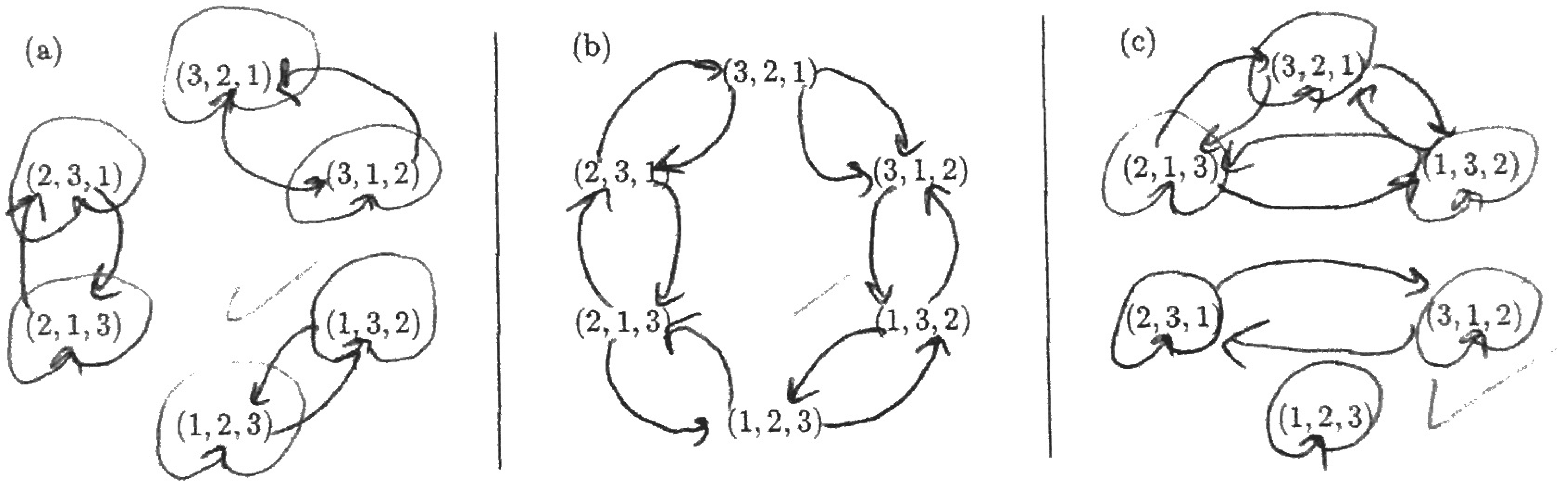
-3

Problem 4. Let $X = \{(1,2,3),(2,1,3),(1,3,2),(2,3,1),(3,1,2),(3,2,1)\}$ be the set of permutations of size 3. For each of these relations R on X, draw its digraph and decide whether each is reflexive, symmetric or transitive (or neither).

(a) $(a_1, a_2, a_3) R (b_1, b_2, b_3)$ if and only if $a_1 = b_1$.

(b) pRq if and only if q can be obtained from p by swapping two adjacent elements. e.g. $(\mathbf{1}, \mathbf{2}, 3)\, R\, (\mathbf{2}, \mathbf{1}, 3)$.

(c) The relation induced from the partition $X_1 = \{(1,2,3)\}, X_2 = \{(2,1,3),(3,2,1),(1,3,2)\}$, $X_3 = \{(2,3,1),(3,1,2)\}$ of X.



a) reflexive, symmetric, and transitive.
Equivalence relation ✓

b) ~~reflexive~~, symmetric, but <u>not</u> transitive

c) reflexive, symmetric, and transitive
Equivalence relation

11/12

$A \cup (\overline{B} \cap \overline{C}) \quad (A \cap \overline{B}) \cup (A \cap \overline{C})$

Problem 5. True or False Circle the answers only **with ink**, next to the questions. No reasoning/calculations will be taken into account.

(a) The sequence $a_n = n! - 2^n$ is decreasing. T. or F.

(b) The sequence $1/\binom{2}{2}, 1/\binom{3}{2}, 1/\binom{4}{2}, \ldots$ is nonincreasing. T. or F.

(c) Given sets $A, B, C \subset U$ the set $A \cup (\overline{B \cup C})$ equals the set $(A \cap \overline{B}) \cup (A \cap \overline{C})$. T. or F.

(d) The name EMMETT has more than 88 rearrangements of its letters. T. or F.

(e) A prime number p divides all the binomial numbers $\binom{p}{1}, \binom{p}{2}, \ldots, \binom{p}{p-1}$. T. or F.

(f) There are more injections than surjections from $\{A, B, C, D\}$ to $\{1, 2, 3, 4\}$. T. or F.

(g) There are more subsets of $\{1, 2, \ldots, 11\}$ of odd size than even size. T. or F.

(h) There are the same nonnegative integer solutions to $x_1+x_2+x_3 = 4$ as positive ingeter solutions to $y_1 + y_2 + y_3 = 7$. T. or F.

$\binom{11}{1} + \binom{3}{3} + \binom{11}{5} + \binom{11}{9} + \binom{11}{8} + \binom{11}{11}$

(i) The coefficient of x^2y^2 in $(x + y + 1)^6$ is $\binom{6}{4}$. T. or F.

(j) There are more symmetric relations than antisymmetric relations on n elements. T. or F.

$\frac{11}{2} \quad \frac{11}{4} \quad \frac{11}{6} \quad 11 \quad 12\ 34\ 8\ 10 \quad C\binom{6}{2} \quad \frac{6!}{4!2!}$

$\binom{6}{2}\binom{4}{2} \quad \frac{6}{2}\ \frac{4}{2} \quad \frac{6!}{2!2!2!} \quad \frac{6 \cdot 5 \cdot 4 \cdot 3}{2 \cdot 2}$

$12\,34\,567 \quad C\binom{9}{2} \quad \frac{6!}{4!2!}\ \frac{4!}{2!2!} \quad \frac{6}{4!2!2!}$

$\frac{6}{4\,2!}$

Symmetric: 2^{n-1}

$\binom{2}{2} \quad \binom{3}{2} \quad \binom{4}{2} \quad 7$

T $\quad \frac{3!}{2!}\ 3 \quad \frac{4!}{2!2!}$

$a_n \qquad a_{n+1}$

$n! - 2^n \;\underline{?}\; (n+1)! - 2^{n+1}$

$n! \;\underline{?}\; (n+1)! - 2^n$

$\frac{p!}{(p-r)!r!}$, $\sin r > 0$

$n \cdot n-1 \cdot n-2$

$(n+1)\,n \qquad n\ 5 \cdot 4 \cdot 3 \cdot 2$

$2 \cdot 2 \cdot 2$

$-n \cdot n! \;\underline{?}\; -2^n$

$0 \;\underline{?}\; n(n)! - 2^n \qquad a_n \leq a_{n+1}$