Problem 1.

(a) Show that

$$1 \cdot 2^1 + 2 \cdot 2^2 + 3 \cdot 2^3 + \cdots + n \cdot 2^n = (n-1) \cdot 2^{n+1} + 2.$$

base case $n=1$ $\quad (1-1)\cdot 2^{1+1} + 2 = 2 = 1 \cdot 2^1$ ✓

inductive step: assume above is true

$1\cdot 2^1 + 2\cdot 2^2 + \ldots + n\cdot 2^n + (n+1)\cdot 2^{n+1} = (n+1-1)\cdot 2^{n+1+1} + 2$

$= (n)\cdot 2^{n+2} + 2$

$= n(2^n \cdot 2^2) + 2$

$= 4 \cdot n \cdot 2^n + 2$ ✓

4/4

$(n-1)\cdot 2^{n+1} + 2 + (n+1)\cdot 2^{n+1}$

$= n\cdot 2^{n+1} - 2^{n+1} + 2 + (n+1)\cdot 2^{n+1}$

$= n\cdot 2^{n+1} + 2^{n+1}(n+1-1) + 2$

$= n\cdot 2^{n+1} + 2^{n+1}(n) + 2$

$= 2n\cdot 2^{n+1} + 2$

$= 2n\cdot 2^n (2) + 2$

$= 4n\cdot 2^n + 2 = 4n\cdot 2^n + 2$

(b) Show by induction that $e^n \geq n+1$ for integers $n \geq 1$ $\quad (e = 2.71\ldots)$.

base case $n=1$ $\quad e^1 = 2.71 > 1+1$

(+1) inductive step:

$e^{n+1} \geq n+1+1 = n+2$

$e \cdot e^n \geq (n+1) + 1$

$e \cdot e^n > e^n \geq (n+1) + 1 > n+1$

Why?

1/4

6

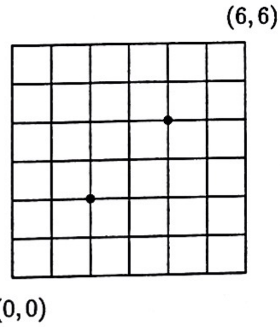
Problem 2. Find the number of grid paths from $(0,0)$ to $(6,6)$ that

(a) go through $(2,2)$,

(b) go through $(4,4)$,

(c) go through $(2,2)$ or $(4,4)$,

(d) do not go through $(2,2)$ ~~and~~ neither one $(4,4)$.



You can write your answers in terms of binomials.

a. $\binom{4}{2}$ $(6-2)(6-2) = (4,4)$

$\to (2,2)$ ways to get from $(0,0)$ to $(2,2)$ • ways to get from $(2,2)$ to $(6,6)$

$$\boxed{\binom{4}{2}\cdot\binom{8}{4}}$$

b. $\boxed{\binom{8}{4}\cdot\binom{4}{2}}$ symmetric with one above

c. ~~intersection~~ goes through both −2

$$\binom{4}{2}\cdot\binom{8}{4} + \binom{8}{4}\binom{4}{2} - \binom{4}{2}$$

$$\boxed{2\binom{4}{2}\binom{8}{4} - \cancel{\binom{4}{2}}}$$

d. all possible $\binom{6+6}{6} = \binom{12}{6}$ ways to get from $(0,0)\to(2,2) = A$

$(2,2)\to(4,4) \Rightarrow B$

$(4,4)\to(6,6) = C$

−2 go through both: $\binom{4}{2}$

$\Sigma \Rightarrow$ all ways

$$\binom{12}{6} - \binom{4}{2}$$

$\Sigma - A - B - C +$ goes through both + goes through either one − ~~intersection~~

$$\binom{4}{2}\cdot\left(\binom{8}{4} - \binom{4}{2}\right)$$

$$\boxed{\binom{12}{6} - \binom{4}{2}\binom{4}{2}\binom{4}{2} - 2\binom{4}{2}\left(\binom{8}{4} - \binom{4}{2}\right)}$$

total — going through both — goes through 1 but not the other for both (2,2) and (4,4) so ×2

Problem 3. Compute the number of 4-subsets of $\{1, 2, 3, \ldots, 10\}$ that:

(a) contain 5,

(b) contain only one prime number $2, 3, 5, 7$,

(c) the minimum or maximum is 5,

(d) the product of three of the entries is 6.

You can write your answers in terms of binomials.

a. 5 _ _ _

9 · 8 · 7   9 numbers left
3 slots left

$\binom{9}{3}$

+10

b. 1 is not listed as a prime number
1, 4, 6, 8, 9, 10 → {6}
6 C 3, 4 ways

$4\binom{6}{3}$

c. min 5 : 6, 7, 8, 9, 10 5 C 3
max 5 : 1, 2, 3, 4 4 C 3
intersection : 0

$\binom{5}{3} + \binom{4}{3}$

~~$|A| + |B| - |A \cap B|$~~

d. $x_1 x_2 x_3 = 6$
$1 \cdot 2 \cdot 3 = 6$ only way
7 numbers left
1 slott left

$\binom{7}{1} = 7$

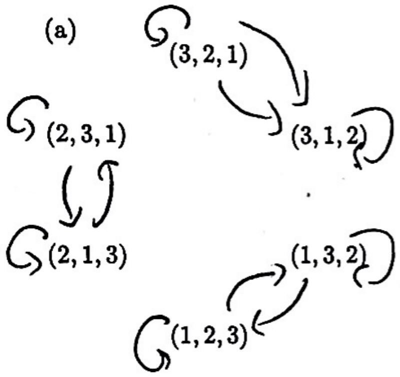
Problem 4. Let $X = \{(1,2,3),(2,1,3),(1,3,2),(2,3,1),(3,1,2),(3,2,1)\}$ be the set of permutations of size 3. For each of these relations R on X, draw its digraph and decide whether each is reflexive, symmetric or transitive (or neither).

(a) $(a_1,a_2,a_3)R(b_1,b_2,b_3)$ if and only if $a_1 = b_1$.

(b) pRq if and only if q can be obtained from p by swapping two adjacent elements. e.g. $(\mathbf{1},\mathbf{2},3)\,R\,(\mathbf{2},\mathbf{1},3)$.

(c) The relation induced from the partition $X_1 = \{(1,2,3)\}, X_2 = \{(2,1,3),(3,2,1),(1,3,2)\}$, $X_3 = \{(2,3,1),(3,1,2)\}$ of X.

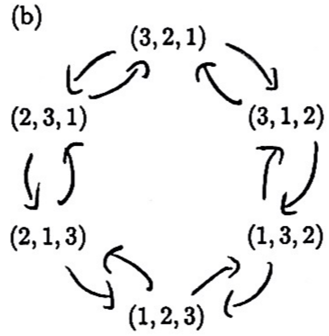
(a)



reflexive
symmetric
transitive

~~meet~~
meets definitions
for all three
properties

(b)

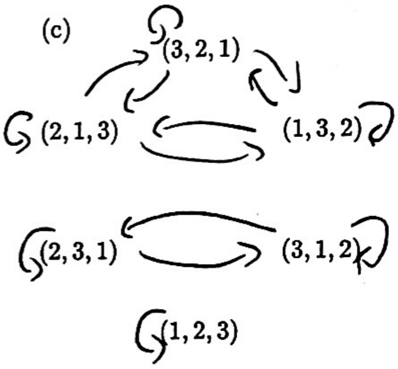


symmetric

not transitive
(3,2,1) R (3,1,2)
(3,1,2) R (1,3,2)
~~(3,2,1) R (1,3,2)~~

not reflexive
~~(3,2,1) R (3,2,1)~~

(c)



reflexive
symmetric
transitive

partitions are inherently
equivalence relations, so must
be all three properties

$\frac{12}{12}$

Problem 5. True or False Circle the answers only **with ink**, next to the questions. No reasoning/calculations will be taken into account.

(a) The sequence $a_n = n! - 2^n$ is decreasing. **T. or F.**

(b) The sequence $1/\binom{2}{2}, 1/\binom{3}{2}, 1/\binom{4}{2}, \ldots$ is nonincreasing. **T. or F.**

(c) Given sets $A, B, C \subset U$ the set $A \cup \overline{B \cup C}$ equals the set $(A \cap \overline{B}) \cup (A \cap \overline{C})$. **T. or F.**

(d) The name EMMETT has more than 88 rearrangements of its letters. **T. or F.**

(e) A prime number p divides all the binomial numbers $\binom{p}{1}, \binom{p}{2}, \ldots, \binom{p}{p-1}$. **T. or F.**

(f) There are more injections than surjections from $\{A, B, C, D\}$ to $\{1, 2, 3, 4\}$. **T. or F.**

(g) There are more subsets of $\{1, 2, \ldots, 11\}$ of odd size than even size. **T. or** ~~**F.**~~

LOL

(h) There are the same nonnegative integer solutions to $x_1+x_2+x_3 = 4$ as positive ingeter solutions to $y_1 + y_2 + y_3 = 7$. **T. or F.**

(i) The coefficient of x^2y^2 in $(x + y + 1)^6$ is $\binom{6}{4}$. **T. or** ~~**F.**~~

(j) There are more symmetric relations than antisymmetric relations on n elements. **T. or F.**

-1 -2 -2 8 $A \cap (\overline{B} \cup \overline{C})$

$\frac{1}{1}$ $\frac{1}{3}$ $\frac{1}{6}$

$A \cup \overline{B} \cap \overline{C}$

4!

x^5y^1
x^4y^2

$\frac{6!}{2 \cdot 2 \cdot 2} = 5! = 24 \cdot 5$ 780

$\binom{5}{2}$ $\frac{5!}{2 \cdot 6} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{n}$

$\binom{6}{2} = \binom{6}{4}$

$(a+b+c)^n$

8