Math 61 Midterm 1

OCT 21 2015

50 minutes

Your Name:

UCLA ID: ______

SECTION: Cross one [illegible]

Sam
1E
1F

Rules: You MUST simplify completely and BO[illegible] answe[illegible]ith an **INK PEN**. You are allowed to use only this paper and pen/pencil. A one-sided hand-written formula sheet is allowed. No calculators, no books, no notebooks, no web access. You MUST write your name and UCLA id. Except for the last problem, you MUST write out your logical reasoning and/or proof in full. You have exactly 50 minutes.

youre

Warning: At 1:50pm your OUTATIME, those caught writing after time get automatic 10% score deduction.

Problem	Value	Score
Problem 1	8	8
Problem 2	10	8
Problem 3	10	7
Problem 4	12	12
Problem 5	10	8
Total	50	43

$\boxed{\text{Proof by induction}}$ Identity: $n2^{n-1} = \sum_{k=0}^{n} \binom{n}{k} k$

Problem 1.

(a) Show that

$$1 \cdot 2^1 + 2 \cdot 2^2 + 3 \cdot 2^3 + \cdots + n \cdot 2^n = (n-1)\cdot 2^{n+1} + 2.$$

$\sum_{k=1}^{n} k2^k = (n-1)\cdot 2^{n+1} + 2$

$\boxed{\text{base case}}$ $n=1$: $1\cdot 2 = 0+2$, $2=2$

$\boxed{\text{Induction step}}$ Assume, prove

$\sum_{k=1}^{n+1} k2^k = n\cdot 2^{n+2} + 2$

$\quad ||$

$\sum_{k=1}^{n} k2^k + (n+1)2^{n+1}$

$\sum_{k=1}^{n} k2^k + (n+1)2^{n+1} = 4n2^n + 2$

$(n-1)2^{n+1} + 2 + (n+1)2^{n+1} = 4n2^n + 2$

$2^{n+1}\left((n-1)+(n+1)\right) = 4n2^n$

4/4 $2^{n+1}(2n) = 4n2^n$

$\boxed{4n2^n = 4n2^n \checkmark}$ $\Rightarrow$ True for all $n \geq 1$ as the problem states

(b) Show by induction that $e^n \geq n+1$ for integers $n \geq 1$ $(e = 2.71\ldots)$.

base case: $n=1$ $e^1 \geq 1+1 = 2$ $\boxed{2.71\ldots > 2 \checkmark}$

induction step: Assume $e^n \geq n+1$, prove $e^{n+1} \geq (n+1)+1 = n+2$

$e^{n+1} = e e^n \geq e(n+1)$ (by assumption)

$\Rightarrow$ $e e^n \geq en + e$

(because $e > 2$ and $2 > 1$)

4/4 $e^{n+1} \geq en + e \geq 2(n+1) = 2n+2 \geq n+2$ $\Rightarrow e^{n+1} \geq n+2$

$\boxed{e^{n+1} \geq n+2 \ \square}$ $\Rightarrow e^n \geq n+1$ for all integers $n \geq 1$

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Problem 2. Find the number of grid paths from $(0,0)$ to $(6,6)$ that

(a) go through $(2,2)$, $(6,6)-(2,2)=(4,4)$

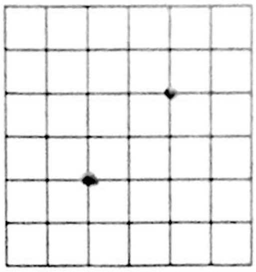
(b) go through $(4,4)$, $(6,6)-(4,4)=(2,2)$

(c) go through $(2,2)$ or $(4,4)$,

(d) do not go through $(2,2)$ and $(4,4)$.

You can write your answers in terms of binomials.

(6,6)



(0,0)

a) Through $(2,2)$: $\binom{2+2}{2}\binom{4+4}{4} = \boxed{\binom{4}{2}\binom{8}{4}}$

b) $\binom{4+4}{4}\binom{2+2}{2} = \boxed{\binom{8}{4}\binom{4}{2}}$

c) Paths through (2,2) + Paths through (4,4) - both

$= \boxed{\binom{4}{2}\binom{8}{4} + \binom{8}{4}\binom{4}{2} - \binom{4}{2}\binom{4}{2}\binom{4}{2}}$

avoid double counting the paths through both (2,2) and (4,4)

d) Total paths: $\binom{6+6}{6} = \binom{12}{6}$

Through both: $\binom{2+2}{2}\binom{2+2}{2}\binom{2+2}{2}$

$= \binom{4}{2}\binom{4}{2}\binom{4}{2}$

Through neither: Total - through both $= \boxed{\binom{12}{6} - \binom{4}{2}\binom{4}{2}\binom{4}{2}}$

Don't go through (2,2) nor (4,4)

556
$\frac{9}{504}$

Problem 3. Compute the number of 4-subsets of $\{1, 2, 3, \ldots, 10\}$ that:

(a) contain 5,

(b) contain only one prime number $2, 3, 5, 7$,   1, 4, 6, 8, 9, 10

(c) the minimum or maximum is 5,

(d) the product of three of the entries is 6.

You can write your answers in terms of binomials.

a) 1 already picked
9 other choices for 2nd, 8 for 3rd, 7 for 4th $\Rightarrow \boxed{\frac{9 \cdot 8 \cdot 7}{3!} = 504}$ −1

b) 1 picked (prime)
6 choices for 2nd, 5 for 3rd, 4 for 4th $\Rightarrow \boxed{4\left(\frac{6 \cdot 5 \cdot 4}{3!}\right) = 120}$ −1

c) 2 sets   +7

5 min (6,7,8,9,10)
1 picked (5)
5 for 2nd, 4 for 3rd, 3 for 4th
$5 \cdot 4 \cdot 3$

5 max (1,2,3,4)
1 picked (5)
4 for 2nd, 3 for 3rd, 2 for 4th
$4 \cdot 3 \cdot 2$ −1

$\boxed{= \frac{5 \cdot 4 \cdot 3}{3!} + \frac{4 \cdot 3 \cdot 2}{3!} = 60 + 24 = 84}$

d) factors of 6 : 1, 2, 3   only 3 options
3 picked, 7 options for 4th $= \boxed{7}$

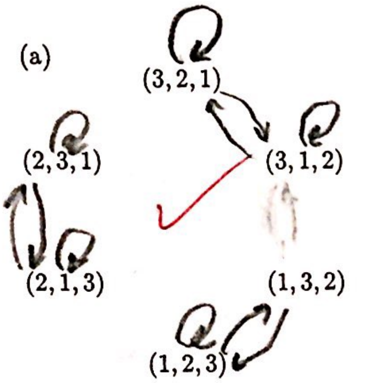
Problem 4. Let $X = \{(1,2,3),(2,1,3),(1,3,2),(2,3,1),(3,1,2),(3,2,1)\}$ be the set of permutations of size 3. For each of these relations R on X, draw its digraph and decide whether each is reflexive, symmetric or transitive (or neither).

(a) $(a_1, a_2, a_3) R (b_1, b_2, b_3)$ if and only if $a_1 = b_1$.

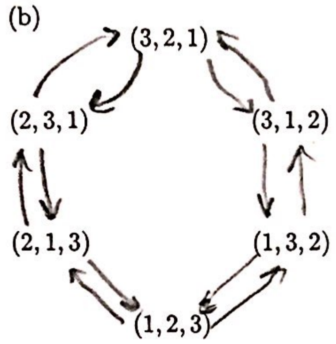
(b) pRq if and only if q can be obtained from p by swapping two adjacent elements. e.g. $(\mathbf{1},\mathbf{2},3)\, R\, (\mathbf{2},\mathbf{1},3)$.

(c) The relation induced from the partition $X_1 = \{(1,2,3)\}, X_2 = \{(2,1,3),(3,2,1),(1,3,2)\}$, $X_3 = \{(2,3,1),(3,1,2)\}$ of X.

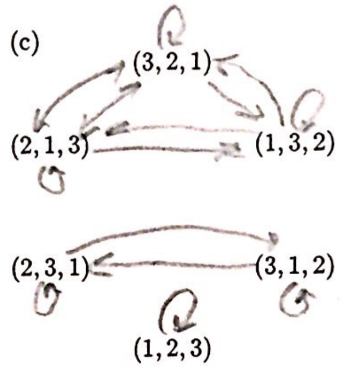
(a)



(b)



(c)



a) reflexive: Yes

Symmetric: Yes ✓

Transitive: Yes (by default)

⇒ Equivalence relation

b) reflexive: No
can't swap

Symmetric: Yes

Transitive: No ✓

c) reflexive: Yes

Symmetric: Yes ✓

Transitive: Yes

It is an equivalence relation bc it partitions

12/12

Problem 5. True or False Circle the answers only **with ink**, next to the questions. No reasoning/calculations will be taken into account.

(a) The sequence $a_n = n! - 2^n$ is decreasing. $n=3$ **T. or F.**

(b) The sequence $1/\binom{2}{2}, 1/\binom{3}{2}, 1/\binom{4}{2}, \ldots$ is nonincreasing. And decreasing **T. or F.**

(c) Given sets $A, B, C \subset U$ the set $A \cup \overline{B \cup C}$ equals the set $(A \cap \overline{B}) \cup (A \cap \overline{C})$. **T. or F.**

(d) The name EMMETT has more than 88 rearrangements of its letters. **T. or F.**

(e) A prime number p divides all the binomial numbers $\binom{p}{1}, \binom{p}{2}, \ldots, \binom{p}{p-1}$. **T. or F.**

(f) There are more injections than surjections from $\{A, B, C, D\}$ to $\{1, 2, 3, 4\}$. **T. or F.**

(g) There are more subsets of $\{1, 2, \ldots, 11\}$ of odd size than even size. 2^{11} subsets **T. or F.**

(h) There are the same nonnegative integer solutions to $x_1+x_2+x_3 = 4$ as positive ingeter solutions to $y_1 + y_2 + y_3 = 7$. **T. or F.** If # of

$\binom{4+3-1}{3-1} = \binom{6}{2}$

(i) The coefficient of x^2y^2 in $(x + y + 1)^6$ is $\binom{6}{4}$. **T. or F.**

(j) There are more symmetric relations than antisymmetric relations on n elements. **T. or F.**

1 2 3

7 sym

1/8 anti

1 2 3

1 2 3 4 5

5 odd

1 2 3 4 even

12 23 3 odd

2 even

1 2 3 1 odd

d) $\frac{6!}{2!2!2!} = \frac{6\cdot5\cdot4\cdot3\cdot2}{8} = 6\cdot5\cdot3 = 90$

c) $A \cup \overline{B \cup C} = A \cup \overline{B} \cap \overline{C} = A \cup \overline{B} \cap A \cup \overline{C}$

$A + \overline{B}\overline{C} = A + \overline{B} \cdot A + \overline{C}$

b) $1, \frac{1}{3}, \frac{1}{12}$

f) $P(4,4) = 4!$ injections

$4\cdot3\cdot2\cdot1 = 4!$ surjections

e) $\binom{7}{1} = 7$, $\binom{7}{2} = \frac{7\cdot6}{2}$

i) $\binom{6}{4} = \frac{6!}{4!2!} = \frac{6\cdot5}{2} = 15$ $\binom{6}{2} = \binom{6}{4}$