Math 61 Midterm 1
OCT 21 2015

tes

Your Name: _____

UCLA ID: _____

SECTION: Cross one box below

Day \ T.A.	John	Zach	Sam
Tuesday	1A	1C	~~1E~~
Thursday	1B	1D	1F

Rules: You MUST simplify completely and BOX all answers with an **INK PEN**. You are allowed to use only this paper and pen/pencil. A one-sided hand-written formula sheet is allowed. No calculators, no books, no notebooks, no web access. You MUST write your name and UCLA id. Except for the last problem, you MUST write out your logical reasoning and/or proof in full. You have exactly 50 minutes.

Warning: At 1:50pm your OUTATIME, those caught writing after time get automatic 10% score deduction.

Problem	Value	Score
Problem 1	8	8
Problem 2	10	10
Problem 3	10	7
Problem 4	12	12
Problem 5	10	8
Total	50	45

$$n \cdot 2^{n+2} + 2$$

Problem 1.

(a) Show that
$$1 \cdot 2^1 + 2 \cdot 2^2 + 3 \cdot 2^3 + \cdots + n \cdot 2^n = (n-1) \cdot 2^{n+1} + 2.$$ ✓

Proceed by induction:

Base case: $S(1)$: $1 \cdot 2^1 = (1-1) \cdot 2^{1+1} + 2$, $2 = 2$ ✓

4/4

Assume $S(n)$, then consider $S(n+1)$

Left hand side: $1 \cdot 2^1 + 2 \cdot 2^2 + \ldots + n \cdot 2^n + (n+1) 2^{n+1}$

substitute based on $S(n) = (n-1) 2^{n+1} + 2 + (n+1) 2^{n+1} = 2^{n+1}(n+1+n-1) + 2$

Now right hand side is correct $= 2n(2^{n+1}) + 2 = n \cdot 2^{n+2} + 2$ ✓

Therefore, since $S(1)$ is true and $S(n+1)$ is true when $S(n)$ is true, then by induction $1 \cdot 2^1 + 2 \cdot 2^2 + \ldots + n \cdot 2^n = (n-1)2^{n+1} + 2$ is true for all $n \geq 1$. ✓

(b) Show by induction that $e^n \geq n+1$ for integers $n \geq 1$ ($e = 2.71\ldots$).

Base Case: $S(1)$: $e^1 \geq 1+1$, $2.71 \geq 2$ ✓
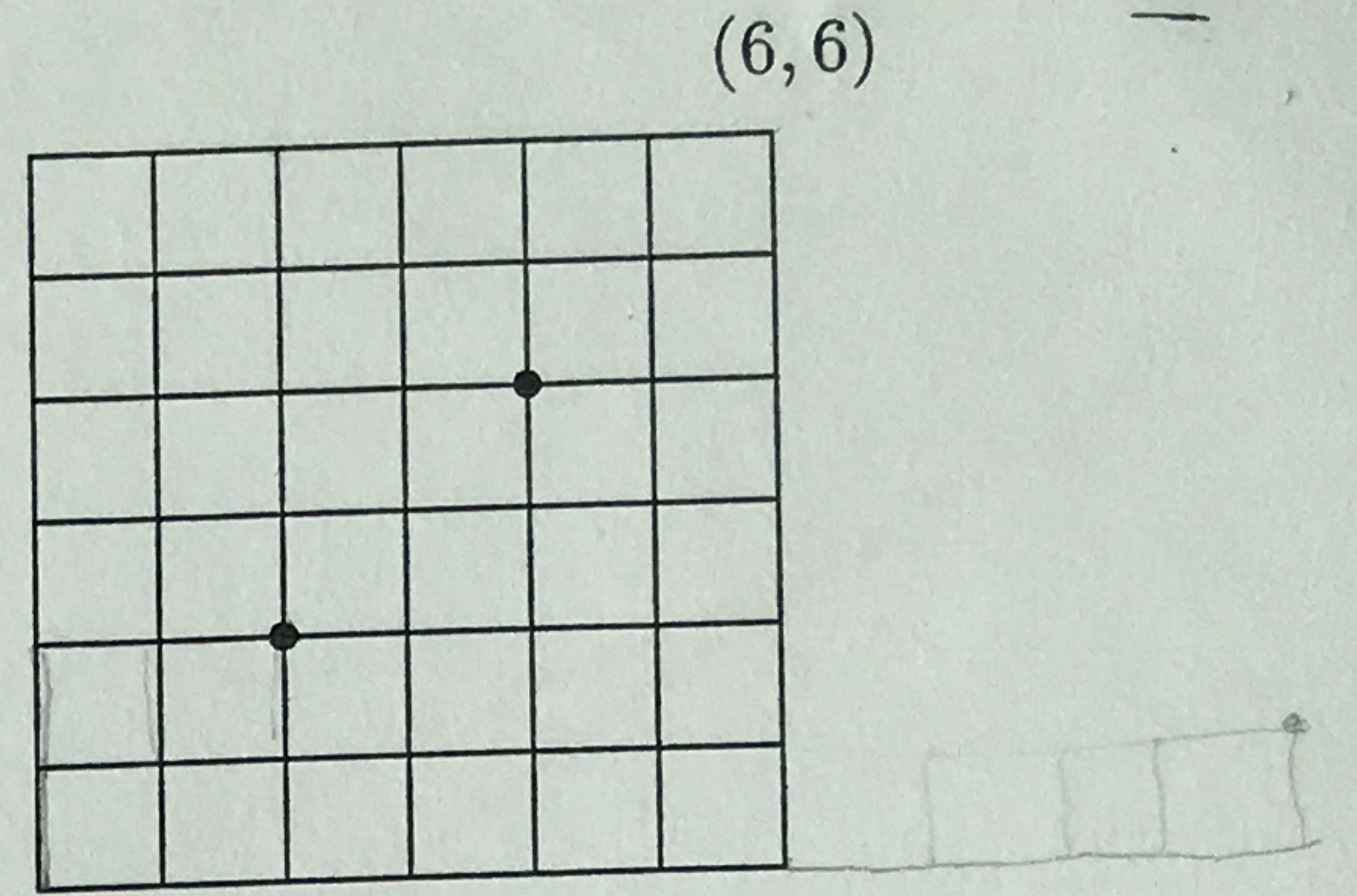
Assume $S(n)$, then consider $S(n+1)$

Left hand side: $e^{n+1} = e \cdot e^n$, from $S(n)$ e^n is at least $n+1$,

4/4 Therefore $e \cdot (n+1) \geq n+2$, $n(e-1) + (e-2) \geq 0$ ✓
True for all $n \geq 1$.

Therefore, since $S(1)$ is true and $S(n+1)$ is true when $S(n)$ is true, then by induction $e^n \geq n+1$ is true for all $n \geq 1$.

Problem 2. Find the number of grid paths from $(0,0)$ to $(6,6)$ that

(a) go through $(2,2)$,

(b) go through $(4,4)$,

(c) go through $(2,2)$ or $(4,4)$,

(d) do not go through $(2,2)$ and $(4,4)$.

You can write your answers in terms of binomials.

$(6,6)$

$(0,0)$

$U R U R$

(a) $\left(\binom{4}{2}\binom{8}{4}\right)$ $\binom{8}{4}\binom{8}{4}$

(b) $\left(\binom{8}{4}\binom{4}{2}\right)$

$\frac{|X| + |Y| - |X \cap Y|}{}$

(c) $\left[\binom{4}{2}\binom{8}{4} + \binom{8}{4}\binom{4}{2} - \binom{4}{2}\binom{4}{2}\binom{4}{2}\right]$

(d) $\left[\binom{12}{6} - \left(\binom{4}{2}\binom{8}{4} + \binom{8}{4}\binom{4}{2} - \binom{4}{2}\binom{4}{2}\binom{4}{2}\right)\right]$

Problem 3. Compute the number of 4-subsets of $\{1, 2, 3, \ldots, 10\}$ that:

(a) contain 5,

(b) contain only one prime number $2, 3, 5, 7$,

(c) the minimum or maximum is 5,

(d) the product of three of the entries is 6.

You can write your answers in terms of binomials.

$\times \; 2$(a) $\quad \dbinom{9}{3}$

$\times \; 2$(b) $\quad \dbinom{4}{1}\dbinom{6}{3}$

$\times \; 7$(c) $\quad \dbinom{4}{3} + \dbinom{5}{3}$

(d) $\quad x_1 \cdot x_2 \cdot x_3 = 6 \qquad C(k+t-1, t-1), \; k=6 \; t=3$

$\dbinom{8}{2}$

-3

correct answer: $\dbinom{7}{1}$

17
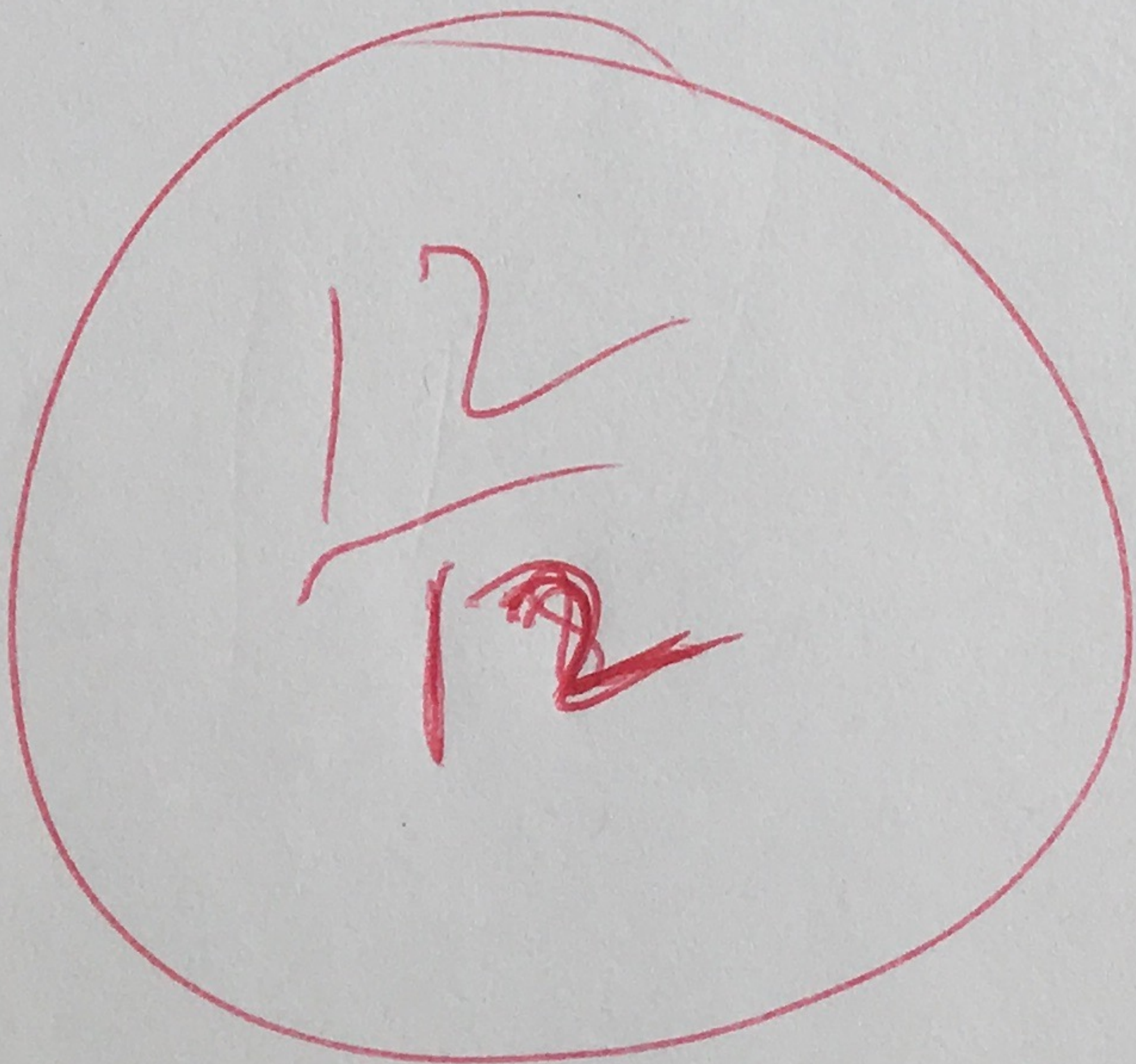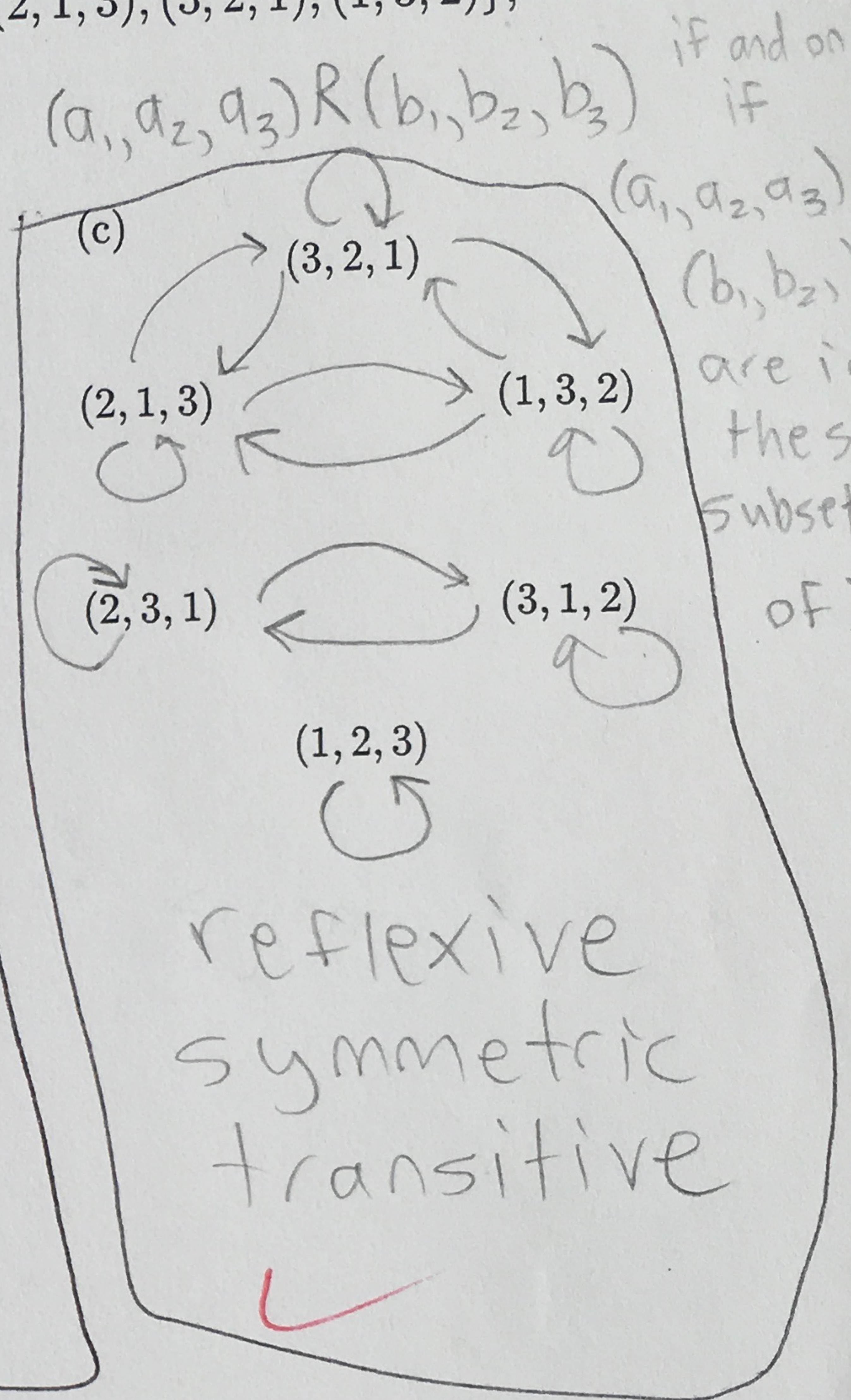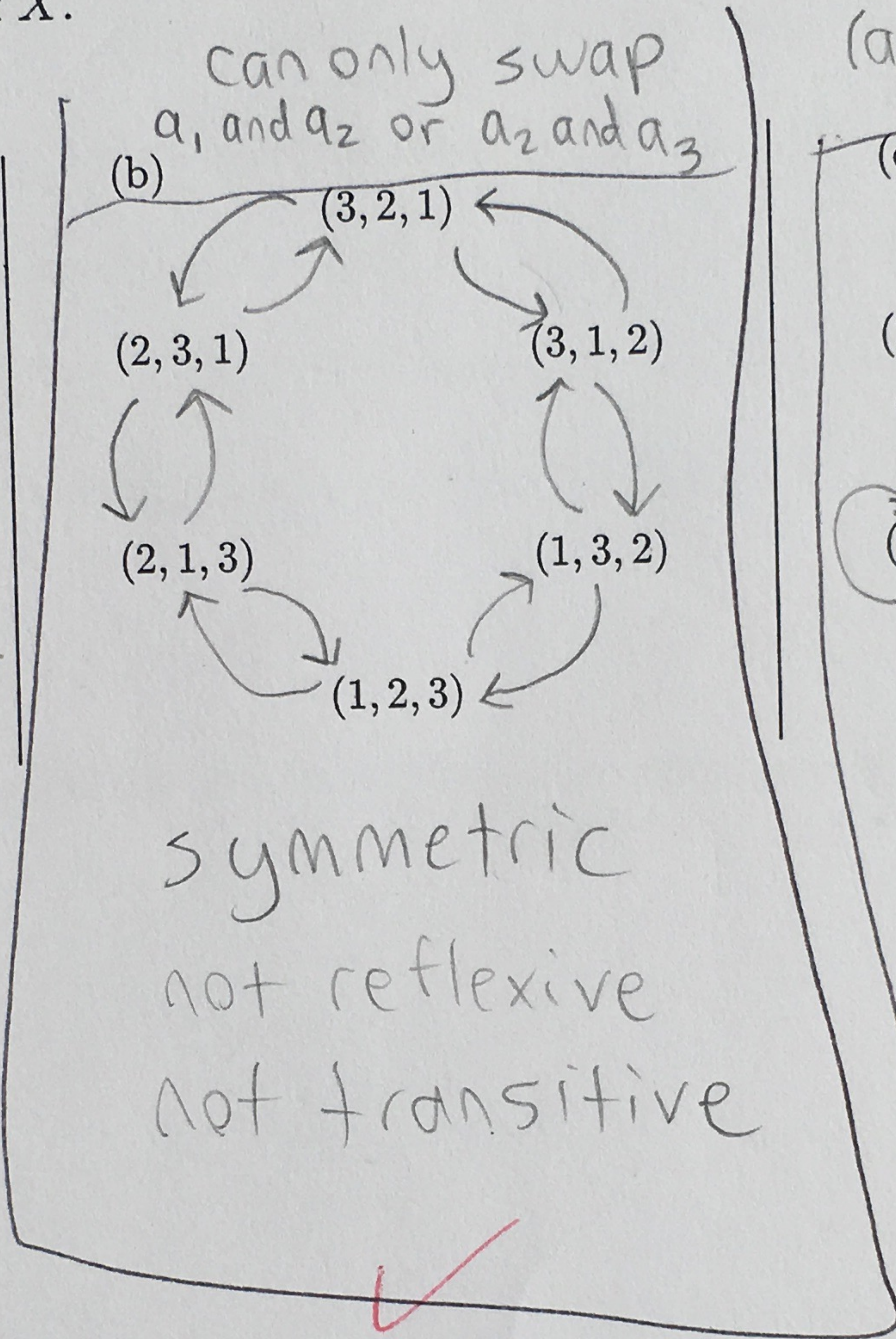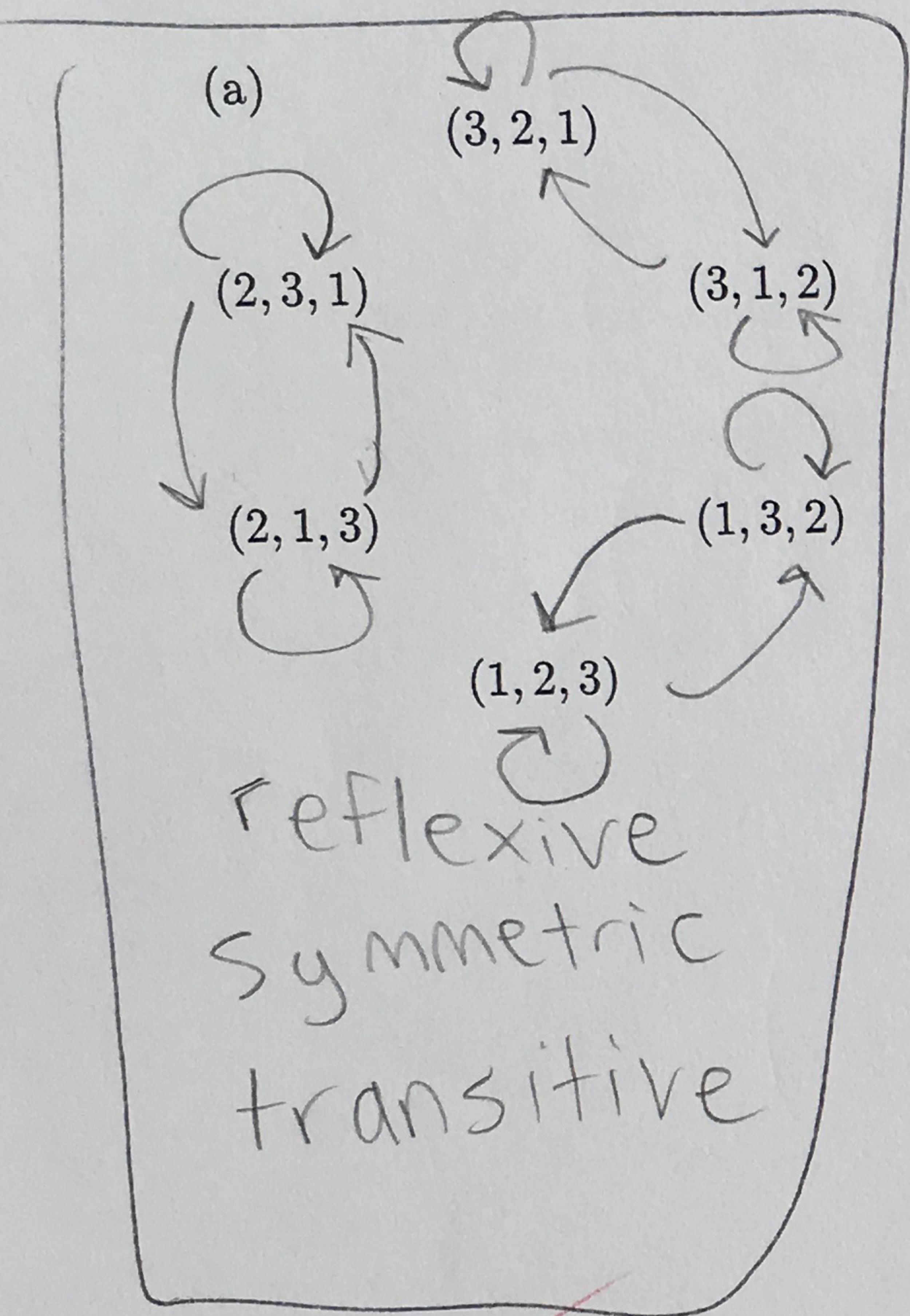
Problem 4. Let $X = \{(1,2,3),(2,1,3),(1,3,2),(2,3,1),(3,1,2),(3,2,1)\}$ be the set of permutations of size 3. For each of these relations R on X, draw its digraph and decide whether each is reflexive, symmetric or transitive (or neither).

(a) $(a_1, a_2, a_3)R(b_1, b_2, b_3)$ if and only if $a_1 = b_1$.

(b) pRq if and only if q can be obtained from p by swapping two adjacent elements.
   e.g. $(1,2,3) R (2,1,3)$.

(c) The relation induced from the partition $X_1 = \{(1,2,3)\}$, $X_2 = \{(2,1,3),(3,2,1),(1,3,2)\}$,
   $X_3 = \{(2,3,1),(3,1,2)\}$ of X.

(a)
(3,2,1)

(2,3,1)   (3,1,2)

(2,1,3)   (1,3,2)

(1,2,3)

reflexive
symmetric
transitive

can only swap
a_1 and a_2 or a_2 and a_3
(b)
(3,2,1)

(2,3,1)   (3,1,2)

(2,1,3)   (1,3,2)

(1,2,3)

symmetric
not reflexive
not transitive

$(a_1, a_2, a_3)R(b_1, b_2, b_3)$ if and only if
if (a_1, a_2, a_3) and (b_1, b_2, b_3) are in the same subset X of X
(c)
(3,2,1)

(2,1,3)   (1,3,2)

(2,3,1)   (3,1,2)

(1,2,3)

reflexive
symmetric
transitive

$\dfrac{12}{12}$

Problem 5. True or False Circle the answers only **with ink**, next to the questions. No reasoning/calculations will be taken into account.

(a) The sequence $a_n = n! - 2^n$ is decreasing. — T. or **F.**

(b) The sequence $1/\binom{2}{2}, 1/\binom{3}{2}, 1/\binom{4}{2}, \ldots$ is nonincreasing. — **T.** or F.

(c) Given sets $A, B, C \subset U$ the set $\overline{A \cup B \cup C}$ equals the set $(A \cap B) \cup (A \cap C)$. — T. or **F.**

(d) The name EMMETT has more than 88 rearrangements of its letters. — **T.** or F.

(e) A prime number p divides all the binomial numbers $\binom{p}{1}, \binom{p}{2}, \ldots, \binom{p}{p-1}$. — **T.** or F.

(f) There are more injections than surjections from $\{A, B, C, D\}$ to $\{1, 2, 3, 4\}$. — T. or **F.**

(g) There are more subsets of $\{1, 2, \ldots, 11\}$ of odd size than even size. — **T.** or F. ✗

(h) There are the same number nonnegative integer solutions to $x_1 + x_2 + x_3 = 4$ as positive integer solutions to $y_1 + y_2 + y_3 = 7$. — **T.** or F.

(i) The coefficient of $x^2 y^2$ in $(x + y + 1)^6$ is $\binom{6}{4}$. — T. or **F.**

(j) There are more symmetric relations than antisymmetric relations on n elements. — **T.** or F. ✗

$x^6 y_1^0 \; x^5 y^4$

$AU(B \cap C)$

$4+3-1, 3-1$

$(6, 2)$

$7+3-1, 3-1$

$\begin{array}{ll} 1-2 & -1 \\ 2-4 & -2 \\ 6-8 & -2 \end{array}$

$24-16$

$\dfrac{2!}{2!\,2!}$

$\dfrac{3}{1!}$