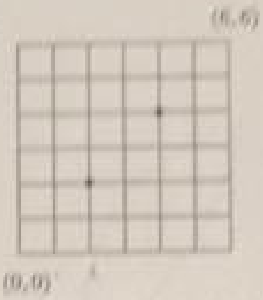
10

Problem 2. Find the number of grid paths from $(0,0)$ to $(6,6)$ that



(a) go through $(2,2)$,

(b) go through $(4,4)$,

(c) go through $(2,2)$ or $(4,4)$,

(d) ~~do not go through $(2,2)$ and $(4,4)$~~.

Ne ther $(2,2)$ NOR $(4,4)$

You can write your answers in terms of binomials.

a) $\boxed{\binom{4}{2}\binom{8}{4}}$

$\binom{4}{2}$ is choosing two "right" steps out of 4 total steps from $(0,0)$ to $(4,4)$

$\binom{8}{4}$ choose 4 rights from 8 total steps

b) $\boxed{\binom{8}{4}\binom{4}{2}}$ same as (a)

c) $\boxed{\binom{4}{2}\binom{8}{4}+\binom{8}{4}\binom{4}{2}-\binom{4}{2}\binom{4}{2}\binom{4}{2}}$

go through both $(2,2)$ and $(4,4)$

$\underbrace{\qquad}$

$\binom{4}{2}\binom{4}{2}\binom{4}{2}$

inclusion excl-

d) don't go through $(2,2)$ and $(4,4)$

$\binom{12}{6}$ total possibilities

(c) of them go through either $(2,2)$ or $(4,4)$

if (c) is A,
(d) wants me to calculate $\bar{A}$ so I compute $U-\bar{A}$

$$\boxed{\binom{12}{6}-\left(\binom{4}{2}\binom{8}{4}+\binom{8}{4}\binom{4}{2}-\binom{4}{2}\binom{4}{2}\binom{4}{2}\right)}$$

Problem 3. Compute the number of 4-subsets of $\{1, 2, 3, \ldots, 10\}$ that:

(a) contain 5,

(b) contain only one prime number 2, 3, 5, 7,

(c) the minimum or maximum is 5,

(d) the product of three of the entries is 6.

You can write your answers in terms of binomials.

+10

a) 4 items that contain 5

9 items $\{1, \ldots 4, 6 \ldots 10\}$ choose 3 to fill in other slots

$\boxed{\binom{9}{3}}$

b) only 1 prime => choose 3 from set of

non primes => $\{1, 4, 6, 8, 9, 10\}$

$\boxed{4 \cdot \binom{6}{3}}$ => for 4 different sets of primes

c) min or max is 5

min is 5

choose 3 from $\{1, 2, 3, 4\}$

max is 5

choose 3 from $\{6, 7, 8, 9, 10\}$

$\boxed{\binom{4}{3} + \binom{5}{3}}$

intersection of these two sets is $\emptyset$

d) prod of 3 entries is 6

must have $\{1, 2, 3\}$ in set ... no other 3 numbers generate a product = 6

choose 1 from set $\{4, \ldots 10\}$

$\binom{7}{1} = \boxed{7}$

Problem 1.

(a) Show that

$S(n)$: $1 \cdot 2^1 + 2 \cdot 2^2 + 3 \cdot 2^3 + \cdots + n \cdot 2^n = (n-1) \cdot 2^{n+1} + 2.$

base case: $n=1$ $\quad 1 \cdot 2^1 = (1-1)2^{1+1} + 2 = 2$

assuming $S(n)$ holds...

inductive step

WTS $\underbrace{1 \cdot 2^1 + 2 \cdot 2^2 + \quad n \cdot 2^n} + (n+1)2^{n+1} = n(2^{n+2}) + 2$

$(n-1) \cdot 2^{n+1} + 2 + (n+1)2^{n+1} = n(2^{n+2}) + 2$

$(2n-2) \cdot 2^n + 2 + (2n+2)2^n = n(2^{n+2}) + 2$

$2^n(2n+2+(2n-2)) + 2 = n(2^{n+2}) + 2$

$2^n(4n) + 2 = n(2^{n+2}) + 2$

$[\ 2^{n+2} \cdot n + 2 = n(2^{n+2}) + 2$

I showed that $S(n)$ implies $S(n+1)$ and $S(1)$ holds. By induction, I show that $S(n)$ holds for all $n \geq 1$

4/4

(b) Show by induction that $e^n \geq n + 1$ for integers $n \geq 1$ $\quad (e = 2.71\ldots)$.

base case: $e^1 \geq 1+1 \quad 2.71.. \geq 2$

$\rightarrow$ assuming $e^n \geq n+1$ holds..

ind step. $\quad e^{n+1} \geq (n+1)+1$

$e(e^n) \geq (n+2)$

$e(n+1) \geq (n+2)$

$\uparrow$

$\rightarrow$ e^n must be at least $n+1$

$en + e \geq n+2$

$\underbrace{\underbrace{(e-1)n}_{(1.71\ldots)n}}_{\text{will always be positive}} + \underbrace{e-2}_{0.71\ldots} \geq 0$ $\Rightarrow$ the sum of two pos[illegible] integers always ≥ 0

$S(n)$ implies $S(n+1)$ and $S(1)$ holds. By induction, I showed that $S(n)$ will hold for all $n \geq 1$.

4/4

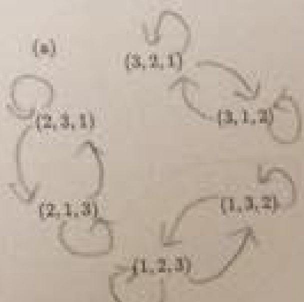
Problem 4. Let $X = \{(1,2,3), (2,1,3), (1,3,2), (2,3,1), (3,1,2), (3,2,1)\}$ be the set of permutations of size 3. For each of these relations R on X, draw its digraph and decide whether each is reflexive, symmetric or transitive (or neither).

(a) $(a_1, a_2, a_3) R (b_1, b_2, b_3)$ if and only if $a_1 = b_1$.

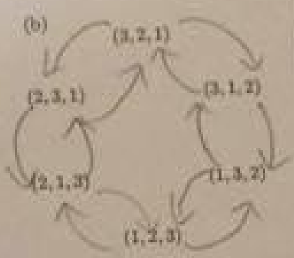
(b) pRq if and only if q can be obtained from p by swapping two adjacent elements. e.g. $(\mathbf{1}, \mathbf{2}, 3) R (\mathbf{2}, \mathbf{1}, 3)$.

(c) The relation induced from the partition $X_1 = \{(1,2,3)\}$, $X_2 = \{(2,1,3), (3,2,1), (1,3,2)\}$, $X_3 = \{(2,3,1), (3,1,2)\}$ of X.

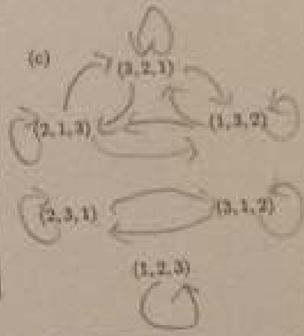
(a)



(b)



(c)



(a) $\boxed{\text{reflexive, symmetric, transitive}}$

formed partitions based on first element

(b) $\boxed{\text{symmetric}}$

not reflexive (no swaps made)

not transitive (swaps different adjacent elements)

(c) $\boxed{\text{reflexive, symmetric, transitive}}$

partitions form equivalence relations,

By definition, these relations will be reflexive, symmetric and transitive

12/12

Math 61 Midterm 1
OCT 21 2015

50 minutes

Your Name: Scott Shi

one box below

Day \ T.A.	John	Zach	Sam
Tuesday	1A	1C	1E
Thursday	1B	1D	1F

You MUST simplify completely and BOX all answers with an **INK PEN**. You are allowe only this paper and pen/pencil. A one-sided hand-written formula sheet is allowed. N tors, no books, no notebooks, no web access. You MUST write your name and UCLA i cept for the last problem, you MUST write out your logical reasoning and/or proof in full. Yo have exactly 50 minutes.

Warning: At 1:50pm your OUTATIME, those caught writing after time get automatic 10% sc deduction.

Problem	Value	Score
Problem 1	8	8
Problem 2	10	10
Problem 3	10	10
Problem 4	12	12
Problem 5	10	10
Total	50	50

10

Problem 5. True or False Circle the answers only **with ink**, next to the questions. No reasoning/calculations will be taken into account.

(a) The sequence $a_n = n! - 2^n$ is decreasing. T. or $\boxed{\text{F.}}$

(b) The sequence $1/\binom{2}{2}, 1/\binom{3}{2}, 1/\binom{4}{2}, \ldots$ is nonincreasing. $\boxed{\text{T.}}$ or F.

(c) Given sets $A, B, C \subset U$ the set $A \cup \overline{B \cup C}$ equals the set $(A \cap \overline{B}) \cup (A \cap \overline{C})$. T. or $\boxed{\text{F.}}$

(d) The name EMMETT has more than 88 rearrangements of its letters. $\boxed{\text{T.}}$ or F.

(e) A prime number p divides all the binomial numbers $\binom{p}{1}, \binom{p}{2}, \ldots, \binom{p}{p-1}$. $\boxed{\text{T.}}$ or F.

(f) There are more injections than surjections from $\{A, B, C, D\}$ to $\{1, 2, 3, 4\}$. T. or $\boxed{\text{F.}}$

(g) There are more subsets of $\{1, 2, \ldots, 11\}$ of odd size than even size. T. or $\boxed{\text{F.}}$

(h) There are the same nonnegative integer solutions to $x_1 + x_2 + x_3 = 4$ as positive ingeter solutions to $y_1 + y_2 + y_3 = 7$. $\boxed{\text{T.}}$ or F.

(i) The coefficient of x^2y^2 in $(x + y + 1)^6$ is $\binom{6}{4}$. T. or $\boxed{\text{F.}}$

(j) There are more symmetric relations than antisymmetric relations on n elements. T. or $\boxed{\text{F.}}$

a) $-1, -2, -2$ F

b) $\frac{1}{1}, \frac{1}{3}, \frac{1}{6} \ldots$ non increasing, not decreasing T

c) $A \cup (\overline{B} \cap \overline{C}) = (A \cup \overline{B}) \cap (A \cup \overline{C})$ F

d) EMMETT $\frac{6!}{2!2!2!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{2} = 90 > 88$ T

e) prime # p divides all $\binom{p}{1}, \binom{p}{2} \ldots$ T p is [illegible] factor

f) more injections than surjections $\{A, B, C, D\}$ to $\{1, 2, 3, 4\}$
4! injections [illegible]
They are equal F

g) more subsets of $\{1, \ldots, 11\}$ of odd size than even $(1-1)$ $\boxed{F}$
$\binom{11}{0}$ vs $\binom{11}{1}$ [illegible]

h) nonnegative $\Rightarrow$ 0 positive T

i) $(x+y+1)^6 \Rightarrow x^2y^2$ $\frac{6!}{2!2!}$ $1 = \frac{6!}{2!4!}$ F

j) more symmetric than antisymmetric on n elements F
Same [illegible] $\Rightarrow$ for [illegible] [illegible]