Math 61
Fall 2017
11/20/17
Time Limit: 50 Minutes

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Day \ T.A.	Riley	Paul	Kevin
Tuesday	1A	1C	1E
Thursday	1B	1D	1F

This exam contains 7 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter your name and SID number on the top of this page, circle your section, and put your initials on the top of every page, in case the pages become separated.

Calculators or computers of any kind are not allowed. You are not allowed to consult any other materials of any kind, including books, notes and your neighbors. You may use the back of this sheet for your notes ("scratch paper"). If you need additional paper, let the proctors know.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a result from class, discussion, or homework you must indicate this** and explain why the result may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	2	2
2	2	2
3	3	3
4	3	2
5	3	3
6	2	2
Total:	15	14

Of course, if you have a question about a particular problem, please raise your hand and one of the proctors will come and talk to you.

1. (2 points) The summand not involving x in the expansion of

$$\left(\frac{x}{2}+\frac{k}{x}\right)^6$$

is 20. What is k?

$$\sum_{k=0}^{n} \binom{n}{k} \frac{x}{2}^{n} \cdot \frac{k}{x}^{n-k}$$

$$\binom{6}{3} = \frac{6!}{3!\,3!} = \quad 5 \cdot 4 = 20$$

$$20 = 20\left(\frac{x}{2}\right)^3 \cdot \left(\frac{k}{x}\right)^3$$

$$20 = 20\left(\frac{k^3}{8}\right)$$

$$\boxed{k = 2}$$

2. (2 points) Show that in the decimal expansion of the quotient of two integers, eventually some block of digits repeats. (Examples:

$$\frac{1}{6} = 0.1\underline{6}66\ldots, \qquad \frac{217}{660} = 0.32\underline{87}8787\ldots)$$

Hint: If we divide a by b, the remainder is one of 1,2,...,b − 1. Consider what happens after b divisions.

2 7

0.285714
7)2.000000

2

2 cases:

Case 1:

clean division & there will be a repetition of zeros.

e.g. $2/5 = 0.2\underline{0}000\ldots$

Case 2:
infinite repetition:

Assume we divide integer a by integer b.
The remainder must be one of $1, 2, \ldots, b-1$.
Each subsequent division of adding a zero to the remainder will result in $1, 2, \ldots, b-1$ as the result still.

pidgeonholes: $1, 2, \ldots, b-1$

pidgeons: after b divisions, you must have come across a repetition of digits, in which case the division will be the same henceforth.

∴ By PHP, the remainder will come to the same digit eventually in which case the block of digits from that number until the number directly before the next instance of the number will repeat infinitely.

3 3. (3 points) Consider the following recurrence relation and initial conditions:

$$U_n = U_{n-1} + 2U_{n-2} + 2n^2 - 10n + 9, \qquad U_1 = 13, \quad U_2 = -12.$$

You may assume that $U_n^{(p)} = -n^2$ is a particular solution to the recurrence relation. Find U_{100}. (No need to simplify your formula.)

$r^2 - r - 2 = 0$ ✓

$(r-2)(r+1) = 0$ ✓

$r_1 = -1, \; r_2 = 2$

$U_n^{(g)} = b(-1)^n + d(2)^n$

$-n^2 = -(n-1)^2 + 2\cdot -(n-2)^2 + 2n^2 - 10n + 9$?

$-n^2 = -n^2 - 2n^2 + 2n^2$

$U_n = U_n^{(g)} + U_n^{(p)}$

$= b(-1)^n + d(2)^n - n^2$

$U_1 = 13 = -b + 2d - 1$

$14 = -b + 2d$

$U_2 = -12 = b + 4d - 4$

$-8 = b + 4d$

$14 = -b + 2d$

$+ \; -8 = b + 4d$

$6 = 6d$

$d = 1$ ✓

$14 = -b + 2$

$b = -12$

$U_n = (-12)(-1)^n + 2^n - n^2$

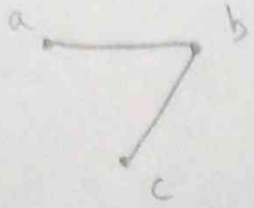
$$\boxed{\begin{aligned} U_{100} &= -12 + 2^{100} - 100^2 \\ &= -10012 + 2^{100} \end{aligned}}$$ ✓

4. (3 points) In each case below, either give an example of a graph satisfying the stated conditions or explain why there is no such example:

   1. a connected graph with 2 edges which does not admit an Euler cycle;
   2. a simple graph with 4 vertices whose adjacency matrix equals its incidence matrix for some ordering of the vertices and the edges.

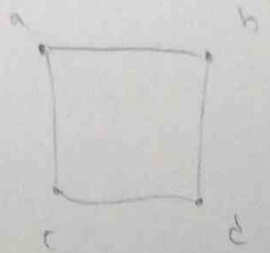
1/1

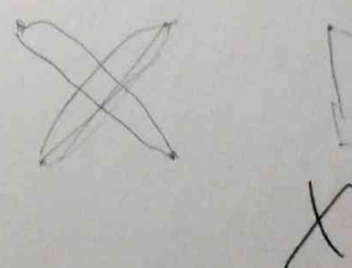
1.



1/2

2.





~~Alternatively must have 4 edges for incidence matrix to be 4 by 4. The only graph w/ (simple) 4 vertices &~~

$$\begin{array}{c|c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \end{array} \qquad \begin{array}{c} a \\ b \\ c \\ d \end{array} \begin{pmatrix} 0 & 1 & 1 & \\ 1 & 0 & & 1 \\ 1 & & 0 & 1 \\ & 1 & 1 & 0 \end{pmatrix}$$

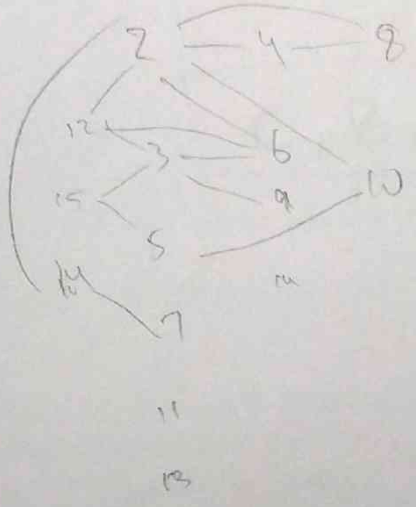
No such example. An adjacency matrix with 4 vertices must be a square matrix with 16 entries. For there to be an incidence matrix equal to the adjacency matrix, it must also be that way. The diagonal of both must be 0's, because no loops are permitted in a simple graph. Properties of the incidence matrix show that the sum of each column must be exactly 2, meaning each column will have 2 one's because a value of 2 isn't permitted on simple graphs. The matrix must also be reflective because of properties of an adjacency matrix. However, should all these properties be true, the graphs of the adjacency matrix & incidence matrix will be distinct

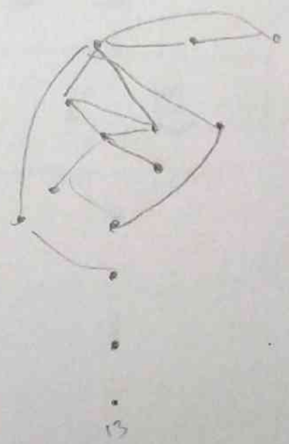
Need more argument.

5. (3 points) In this problem, we are concerned with the divisibility relation on the set of integers $\{2, 3, 4, \ldots\}$ given by

$$\{(x, y) \mid x \text{ divides } y\}.$$

Consider the associated digraph, and let G be the underlying (undirected) graph, obtained by forgetting that the edges have a direction. Is G connected? What is the shortest path length for two vertices in the same component? *Hint: the answer to the last question will depend on the two vertices picked.*

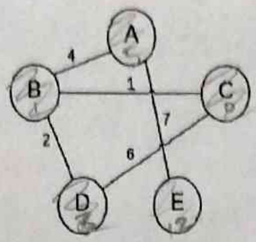




Yes, G is connected because any 2 vertices will be connected by their least common multiple.

The shortest path length for 2 vertices in the same component is 1 if one number divides the other, or 2 if neither is divisible by the other (through the LCM).

6. (2 points) Write the order in which the shortest path algorithm visits the vertices of the graph when finding the shortest path from C to E. (Here, we say that the algorithm *visits* a vertex when the label of the vertex becomes permanent.)



$C \to E$

$\boxed{C, B, D, A, E}$