MT2

Math 61
Fall 2017
11/20/17
Time Limit: 50 Minutes

[redacted]

Day \ T.A.	Riley	Paul	Kevin
Tuesday	1A	1C	1E
Thursday	1B	1D	1F

This exam contains 7 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter your name and SID number on the top of this page, circle your section, and put your initials on the top of every page, in case the pages become separated.

Calculators or computers of any kind are not allowed. You are not allowed to consult any other materials of any kind, including books, notes and your neighbors. You may use the back of this sheet for your notes ("scratch paper"). If you need additional paper, let the proctors know.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a result from class, discussion, or homework you must indicate this** and explain why the result may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	2	2
2	2	1
3	3	3
4	3	2
5	3	3
6	2	2
Total:	15	13

Of course, if you have a question about a particular problem, please raise your hand and one of the proctors will come and talk to you.

1. (2 points) The summand not involving x in the expansion of - constant term

$$\left(\frac{x}{2}+\frac{k}{x}\right)^6$$

not involving x

is 20. What is k?

binomial coefficients

$(a+b)^n = \sum_{k=0}^{n} C(n,k)\, a^{n-k} b^k$

$(\frac{1}{2}x + kx^{-1})^6$

$(a+b)^6$

$a = \frac{1}{2}x \quad b = \frac{k}{x}$

$C(6,0)(\frac{1}{2}x)^6 \cancel{(\frac{k}{x})^0} + C(6,1)(\frac{1}{2}x)^5(\frac{k}{x})^1 + C(6,2)(\frac{1}{2}x)^4(\frac{k}{x})^2 + C(6,3)(\frac{1}{2}x)^3(\frac{k}{x})^3$

$+ C(6,4)(\frac{1}{2}x)^2(\frac{k}{x})^4 + C(6,5)(\frac{1}{2}x)^1(\frac{k}{x})^5 + C(6,6)\cancel{(\frac{1}{2}x)^0}(\frac{k}{x})^6$

want constant --- no x

$C(6,3)(\frac{1}{2}x)^3(\frac{k}{x})^3$

$\frac{6!}{3!3!} \cdot \frac{\cancel{x^3}}{8} \cdot \frac{k^3}{\cancel{x^3}} \Rightarrow \frac{\cancel{6}\cdot 5 \cdot \cancel{4}\, k^3}{3 \cdot 2 \cdot \cancel{4} \cdot \cancel{8}\, 2} = 20$

$\frac{\cancel{4}(5)(k^3)}{\cancel{4}\, 2} = 20$

$5k^3 = 40$

$k^3 = \frac{40}{5}$

$k = \sqrt[3]{\frac{40}{5}}$

$\frac{\cancel{3}\cancel{(4)}(5)\cancel{(4)}}{\cancel{(3)}\cancel{(2)}} \cdot \frac{1}{\cancel{8}\,2} \cdot k^3 = 20$

$\frac{(5)k^3}{2} = 20$

$5k^3 = 40$

$\frac{2\,\cancel{(6)}(5)\cancel{(4)}}{\cancel{(3)}\cancel{(2)}} \cdot \frac{1k^3}{\cancel{8}}$

2. (2 points) Show that in the decimal expansion of the quotient of two integers, eventually some block of digits repeats. (Examples:

$$\frac{1}{6} = 0.1\underline{6}66\ldots, \qquad \frac{217}{660} = 0.32\underline{87}8787\ldots)$$

Hint: If we divide a by b, the remainder is one of 1,2,...,b − 1. Consider what happens after b divisions.

pigeon hole

$\frac{x}{b}$ $\underbrace{0\%b \quad 1\%b \quad 2\%b \quad 3\%b \quad 4\%b \quad 5\%b}$ } pigeon holes (b−1)

$\frac{x}{b}$ falls into 1 of these buckets

if we divide b times (b pigeons) then some pigeon hole will have 2 pigeons by PHP.

if we then keep dividing, the pigeons will fall into their respective pigeon holes in the same order ?

like the pattern will be repeated forever

Remainder obtained determines the subsequent digits in the expansion

↳ if you obtain the same 2 remainders, the block of digits will repeat forever

case 1: 0%b will always be remainder 0

1 ~ 5%b

3

3. (3 points) Consider the following recurrence relation and initial conditions:

$$U_n = \underbrace{U_{n-1} + 2U_{n-2}}_{\text{homog.}} + \underbrace{2n^2 - 10n + 9}_{\text{inhomog.}}, \qquad U_1 = 13, \quad U_2 = -12.$$

You may assume that $U_n^{(p)} = -n^2$ is a particular solution to the recurrence relation. Find U_{100}. (No need to simplify your formula.)

$t^n = t^{n-1} + 2t^{n-2}$

$t^n - t^{n-1} - 2t^{n-2} = 0$

$t^2 - t - 2 = 0$

$(t-2)(t+1) = 0$

$t = 2, -1$ ✓

$u_n = (b)(2^n) + (d)(-1)^n$

$u_n = cn^2 + en + f$

$cn^2 + en + f = [c(n-1)^2 + e(n-1) + f] + 2[c(n-2)^2 + e(n-2) + f]$
$+ 2n^2 - 10n + 9$

$cn^2 + en + f = [c(n^2 - 2n + 1) + en - e + f]$
$+ 2[c(n^2 - 4n + 4) + en - 2e + f]$
$+ 2n^2 - 10n + 9$

$cn^2 + en + f = [cn^2 - 2cn + c + en - e + f] + 2[cn^2 - 4cn + 4c + en - 2e + f]$
$+ 2n^2 - 10n + 9$

$0 = [cn^2 - 2cn + c + en - e + f] + [2cn^2 - 8cn + 8c + 2en - 4e + 2f] + 2n^2 - 10n + 9$
$- cn^2 - en - f$

$n^2[c + 2c - c + 2] + n[\underbrace{-2c + e - 8c + 2e - 10 - e}] + [\underbrace{c - e + f + 8c - 4e + 2f - f + 9}]$

$c + 2c - c + 2 = 0$

$2c + 2 = 0$

$2c = -2$

$c = -1$

$(-2)(-1) + e - 8(-1) + 2e - 10 - e = 0$

$2 + e + 8 + 2e - 10 - e = 0$

$2e = 0$

$e = 0$

$(-1) - 0 + f + 8(-1) - 0 + 2f - f + 9 = 0$

$-1 - 8 + 2f + 9 = 0$

$-9 + 9 + 2f = 0$

$f = 0$

$u_n = b(2^n) + d(-1)^n - n^2$

$u_1 = 13$

$u_2 = -12$

$13 = 2b - d - 1$

$14 = 2b - d$

$d = 2b - 14$

$d = 2 - 14$

$d = -12$

$-12 = 4b + d - 4$

$-12 = 4b + (2b - 14) - 4$

$-12 = 6b - 18$

$6 = 6b$

$1 = b$

$u_n = 2^n + (-12)(-1)^n - n^2$ ✓

$u_{100} = 2^{100} + (-12)(-1)^{100} - (100)^2$

4. (3 points) In each case below, either give an example of a graph satisfying the stated conditions or explain why there is no such example:

   1. a connected graph with 2 edges which does not admit an Euler cycle;
   2. a simple graph with 4 vertices whose adjacency matrix equals its incidence matrix for some ordering of the vertices and the edges.

1/1

1. 

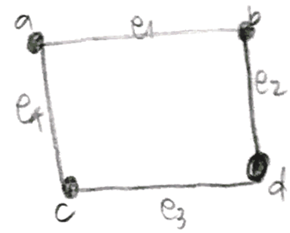
2 edges ✓
connected ✓
no Euler b/c
2 vertices have
odd degree (1)

2.

	a	b	c	d
a	0	1	1	0
b	1	0	0	1
c	1	0	0	1
d	0	1	1	0

so 4 edges

	e_1	e_2	e_3	e_4
a	1	0	0	1
b	1	1	0	0
c	0	0	1	1
d	0	1	1	0



need 2 in each column

2 in each row

1/2

Not possible.

Correct example of adjacency and incidence matrix.

	a	b	c	d
a	0	1	1	
b	1	0		
c	1		0	
d				0

- diagonal 0s since it's simple
- Need 2 1s in each column since also edge (incidence matrix)

Then vertex a incident on vertex b & c example (adj matrix)

But edge says b & c connected

- we have a triangle & 1 more vertex in our graph

(contradiction b/c sum of columns needs to be 2

simple graph w/ 4 vertices whose adjacency matrix equals incidence matrix for some ordering of vertices & edges
↳ 4x4 matrix
↳ Each col. adds up to 2

	a	b	c	d
a	0	0	1	1
b	0	0		
c	1	1	0	
d	1	1		0

// edges

	a	b	c	d
a	0	1	0	1
b	1	0	1	
c	0		0	
d	1		1	0

// edges

	a	b	c	d
a	0	1	1	0
b	1	0		1
c	1		0	1
d	0			0

// edges

5. (3 points) In this problem, we are concerned with the divisibility relation on the set of integers $\{2, 3, 4, \ldots\}$ given by

$$\{(x, y) \mid x \text{ divides } y\}.$$

Consider the associated digraph, and let G be the underlying (undirected) graph, obtained by forgetting that the edges have a direction. Is G connected? What is the shortest path length for two vertices in the same component? *Hint: the answer to the last question will depend on the two vertices picked.*

set of integers
{ 2, 3, 4, 5 ..... ∞}

adjacent to all multiples of 2

adjacent to all multiples of 3

etc...

- G is connected b/c if we take all integers, that's an infinite set and lets say we're looking at a prime number - no factors! but if we then take multiples of the prime number, then it'll be divisible by 2. for example, 7 is prime (7)(2)=14
7 is adjacent to 14 which is adjacent to 2

- it is connected b/c if # isnt prime you can make make the prime factor tree and then it'll be adjacent to prime numbers (2, 3, etc...)

- ∞ set so any vertex m can connect to n by having another vertex (m)(n) be adjacent to both m and n

lol sorry for my terrible explanation

shortest path between 2 vertices in same component

- if x divides y or y divides x then path length of shortest path is 1
example 2 divides 4 (direction doesnt matter)

2 —— 4 path length 1

- if x doesnt divide y and y doesnt divide x
path length is 2 for shortest path
b/c x is adjacent to (x)(y) which is adjacent to y

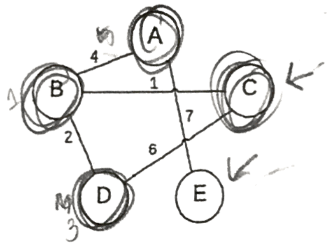
example 3 and 5

path cant be length 1 b/c
3 doesnt divide 5
5 doesnt divide 3
next shortest length is 2

6. (2 points) Write the order in which the shortest path algorithm visits the vertices of the graph when finding the shortest path from C to E. (Here, we say that the algorithm *visits* a vertex when the label of the vertex becomes permanent.)

Dijkstra's algorithm

C → E



$x = C$
$L(C) = 0$
$P = \{C\}$
~~$L(B) = 1$~~
~~$L(D) = 6$~~

$x = B$
$L(B) = 1$
$P = \{C, B\}$
$L(A) = 5$
~~$L(D) = 3$~~

$x = D$
$L(D) = 3$
$P = \{C, B, D\}$

$x = A$
$L(A) = 5$
$L(E) = 5 + 7 = 12$

$P = \{C, B, D, A, E\}$
$x = E$

order of vertex visits in algorithm
C, B, D, A, E