Math 61
Fall 2017
10/23/17
Time Limit: 50 Minutes

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SID Number: 504 993 197

Day \ T.A.	Riley	Paul	Kevin
Tuesday	1A	1C	1E
Thursday	1B	1D	1F

This exam contains 7 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter your name and SID number on the top of this page, and put your initials on the top of every page, in case the pages become separated. Also, have your photo ID on the desk in front of you during the exam.

Calculators or computers of any kind are not allowed. You are not allowed to consult any other materials of any kind, including books, notes and your neighbors. You may use the back of this sheet for your notes ("scratch paper"). If you need additional paper, let the proctors know.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a result from class, discussion, or homework you must indicate this** and explain why the result may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Of course, if you have a question about a particular problem, please raise your hand and one of the proctors will come and talk to you.

Problem	Points	Score
1	2	2
2	3	3
3	2	2
4	3	3
5	2	2
6	3	3
Total:	15	15

1. (2 points) Let A and B be sets. Does the following identity hold?

$$\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$$

If it holds, provide a proof. If it doesn't hold, give a counter-example. (Here, $\mathcal{P}(X)$ denotes the power set of X.)

No,

Let $A = \{1,2,3\}$ and $B = \{2,4,6\}$

The set $\{1,2,4\}$ is not in $P(A)$ or in $P(B)$, but is in $P(A \cup B)$

$P(A \cup B) \supseteq P(A) \cup P(B)$, but
$P(A \cup B) \not\subseteq P(A) \cup P(B)$

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2. (3 points) Prove that

$$\frac{1}{2}+\frac{2}{3}+\cdots+\frac{n}{n+1}<\frac{n^2}{n+1}$$

for all integers $n \geq 2$.

Base case $n=2$

$\frac{1}{2}+\frac{2}{3} < \frac{2^2}{2+1}$ $\qquad \frac{1}{2} < \frac{2}{3}$

$\frac{7}{6} < \frac{4}{3}$ ✓ $\qquad 3 < 4$

I-Hop. - Assume $\frac{1}{2}+\frac{2}{3}+\cdots+\frac{n}{n+1} < \frac{n^2}{n+1}$ is true

WTS: $\underbrace{\frac{1}{2}+\frac{2}{3}+\cdots+\frac{n}{n+1}}+\frac{n+1}{n+2} < \frac{(n+1)^2}{n+2}$

By I-hop, $\frac{1}{2}+\frac{2}{3}+\cdots+\frac{n}{n+1} < \frac{n^2}{n+1}$

$\frac{n^2}{n+1}+\frac{n+1}{n+2} < \frac{(n+1)^2}{n+2}$

$\frac{n^2(n+2)+(n+1)(n+1)}{(n+1)\cancel{(n+2)}} < \frac{(n+1)^2}{\cancel{n+2}}$

$n^2(n+2)+(n+1)(n+1) < (n+1)^3$

$n^3+2n^2+n^2+2n+1 < (n^2+2n+1)(n+1)$

$n^3+3n^2+2n+1 < n^3+2n^2+n+n^2+2n+1$

$n^3+3n^2+2n+1 < n^3+3n^2+3n+1$

$0 < n$

3

for any $n \geq 2$ ✓

∴ By principle of mathematical induction,

$\frac{1}{2}+\frac{2}{3}+\cdots+\frac{n}{n+1} < \frac{n^2}{n+1}$ for all integers $n \geq 2$

3. (2 points) Let s be the sequence

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$$s_n = \prod_{i=1}^{n} i^{(-1)^i}$$

for all positive integers n. What are s_1, s_2, s_3, s_4? Is s increasing? Decreasing? Nonincreasing? Nondecreasing?

$$s_1 = \prod_{i=1}^{1} i^{(-1)^i} = 1^{(-1)^1} = 1$$

$$s_2 = \prod_{i=1}^{2} i^{(-1)^i} = 2^{(-1)^2} \cdot s_1 = 2$$

$$s_3 = \prod_{i=1}^{3} i^{(-1)^i} = 3^{(-1)^3} \cdot s_2 = \frac{2}{3}$$

$$s_4 = \prod_{i=1}^{4} i^{(-1)^i} = 4^{(-1)^4} \cdot s_3 = \frac{8}{3}$$

s is not increasing

'' decreasing

'' nonincreasing

'' nondecreasing

4. (3 points) For this problem recall that a function is a particular type of relation. Let X be a set of n elements ($n \geq 1$). Describe all functions $f : X \to X$ which are also equivalence relations. For each such function, determine the number of equivalence classes.

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$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ \vdots \\ n \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 2 \\ 3 \\ \vdots \\ n \end{pmatrix}$$

R

Equivalence relation: reflexive, symmetric & transitive

reflexive: $\forall x \in X$, (x,x) must be in R

↳ function must map to itself

A function is an equivalence relation if & only if each x in the domain maps to itself.
If $a = b$, then $f(a) = f(b)$, so each element in the domain can only map to one element, namely itself.

The number of equivalence classes for such a function is n because every pair (x,x) for any $x \in X$ defines its own equivalence class.

5. (2 points) Two dice are rolled, one blue and one red. How many outcomes have either the blue die 3 or an even sum or both?

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By inclusion exclusion principle:

$|A|$ = blue die 3

$|B|$ = even sum

$|A \cup B| = |A| + |B| - |A \cap B|$

$|A|$ = 1 possibility for blue die
6 possibilities for red die
M.P. $\rightarrow 1 \times 6 = 6$

$|B|$ = case 1 odd — odd
case 2 or even — even

so case 1: $3 \times 3 = 9$
case 2: $3 \times 3 = 9$

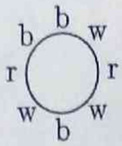
A.P. $\rightarrow 9 + 9 = 18$

$|A \cap B|$ = blue die must be 3 $\rightarrow$ 1 possibility
red die must be 1, 3, or 5 for sum to be even
$\hookrightarrow$ 3 possibilities

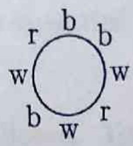
MP: $1 \times 3 = 3$

$|A \cup B| = 6 + 18 - 3 = \boxed{21 \text{ outcomes}}$

6. (3 points) There are 2 red balls, 3 white balls, and 3 blue balls. It is understood that balls of the same color are indistinguishable. The balls are to be arranged in a circle, for example like this:



(a) In how many ways can they be arranged if we don't distinguish between arrangements obtained from each other by rotation? I.e. we don't distinguish for example between the arrangement above and the following one:



(b) What if we add the condition that the red balls are to be placed next to each other?

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a. 2 red, 3 white, 3 blue

$$\frac{\#\text{ of distinguishable arrangements}}{8} = \#\text{ of indistinguish. arrangements}$$

$$\frac{\#\text{ of dist.}}{8} = \frac{\frac{8!}{3!3!2!}}{8} = \boxed{\frac{8!}{8\cdot 3!3!2!}}$$

b. red balls next to each other
↳ group the red balls together & assume there's 7 seats.
1 red, 3 white, 3 blue

using the same logic as part a.

$$\boxed{\frac{7!}{7\cdot 3!\,3!}}$$

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