MT 1

Math 61
Fall 2017
10/23/17
Time Limit: 50 Minutes

[redacted]

Day \ T.A.	Riley	Paul	Kevin
Tuesday	1A	1C	1E
Thursday	1B	1D	1F

in Bunche

This exam contains 7 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter your name and SID number on the top of this page, and put your initials on the top of every page, in case the pages become separated. Also, have your photo ID on the desk in front of you during the exam.

Calculators or computers of any kind are not allowed. You are not allowed to consult any other materials of any kind, including books, notes and your neighbors. You may use the back of this sheet for your notes ("scratch paper"). If you need additional paper, let the proctors know.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a result from class, discussion, or homework you must indicate this** and explain why the result may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Of course, if you have a question about a particular problem, please raise your hand and one of the proctors will come and talk to you.

Problem	Points	Score
1	2	1
2	3	3
3	2	2
4	3	2
5	2	2
6	3	3
Total:	15	13

[redacted]

1. (2 points) Let A and B be sets. Does the following identity hold?

$$\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$$

If it holds, provide a proof. If it doesn't hold, give a counter-example. (Here, $\mathcal{P}(X)$ denotes the power set of X.)

Make sure LHS = RHS

$\boxed{\text{Following identity holds}}$

Start with LHS
P(A ∪ B)

Lets say Z ∈ P(A∪B)
then Z ⊆ (A ∪ B) 40
Z ⊆ A or Z ⊆ B
Z ∈ P(A) or Z ∈ P(B)
so... Z ∈ P(A) ∪ P(B) ← this is RHS

Start with RHS
P(A) ∪ P(B)
Lets say Z ∈ P(A) ∪ P(B)
Z ∈ P(A) or P(B)
Z ⊆ A or B
Z ∈ P(A ∪ B) ← LHS

.2

This doesnt hold :(

P(A∪B) = P(A) ∪ P(B)

- power set of A+B
- these are defined before you combine a and b
- subset could include an element from a & an element from b

Ex: A = {1} P(A) = ∅, {1}
B = {2} P(B) = ∅, {2}

P(A∪B) = ∅, {1}, {2}
{1, 2}

[redacted]

2. (3 points) Prove that

$$\underbrace{\frac{1}{2}+\frac{2}{3}+\cdots+\frac{n}{n+1}<\frac{n^2}{n+1}}_{S(n)}$$

for all integers $n \geq 2$.

- Base case $n=2$

$\frac{1}{2}+\frac{2}{3}<\frac{4}{3}$

$\frac{3}{6}+\frac{4}{6}<\frac{8}{6}$

$\frac{7}{6}<\frac{8}{6}$ ✓

- Inductive step

assume $S(n)$

show $S(n+1)$

assume: $\frac{1}{2}+\frac{2}{3}+\dots+\frac{n}{n+1}<\frac{n^2}{n+1}$

Show: $\underbrace{\frac{1}{2}+\frac{2}{3}+\dots+\frac{n}{n+1}+\frac{n+1}{n+2}}_{\text{LHS}}<\underbrace{\frac{(n+1)^2}{n+2}}_{\text{RHS}}$

start w/ assumption

$\frac{1}{2}+\frac{2}{3}+\dots+\frac{n}{n+1}<\frac{n^2}{n+1}$

$\frac{1}{2}+\frac{2}{3}+\dots+\frac{n}{n+1}+\frac{n+1}{n+2}<\frac{n^2}{n+1}+\frac{n+1}{n+2}$ add $\left(\frac{n+1}{n+2}\right)$ to both sides

" $<\frac{(n^2)(n+2)+(n+1)(n+1)}{(n+1)(n+2)}$

" $<\frac{n^3+2n^2+n^2+2n+1}{(n+1)(n+2)}$

" $<\frac{n^3+3n^2+2n+1}{(n+1)(n+2)}$ ← if we prove that this is less than RHS then LHS < RHS as well

" $<\frac{n^3+3n^2+2n+1}{(n+1)(n+2)}<\frac{(n+1)^2}{n+2}$

" $<\frac{n^3+3n^2+2n+1}{(n+1)(n+2)}<\frac{(n^2+2n+1)}{n+2}\cdot\frac{(n+1)}{n+1}$

" $<\frac{n^3+3n^2+2n+1}{(n+1)(n+2)}<\frac{n^3+n^2+2n^2+2n+n+1}{(n+2)(n+1)}$ 3

" $<\underbrace{\frac{n^3+3n^2+2n+1}{(n+1)(n+2)}}_{\text{LHS}}<\underbrace{\frac{n^3+3n^2+3n+1}{(n+2)(n+1)}}_{\text{RHS}}$ for $n \geq 2$ ✓

∴ $S(n)$ is true for $n \geq 2$

2

3. (2 points) Let s be the sequence

PEMDAS

$$s_n = \prod_{i=1}^{n} i^{(-1)^i}$$

for all positive integers n. What are s_1, s_2, s_3, s_4? Is s increasing? Decreasing? Nonincreasing? Nondecreasing?

$S_1 = \prod_{i=1}^{1} i^{(-1)^i}$

$S_1 = 1^{(-1)^1}$

$S_1 = 1^{(-1)}$

$S_1 = \frac{1}{1^1}$

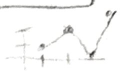
$S_1 = \boxed{1}$

$S_2 = \prod_{i=1}^{2} i^{(-1)^i}$

$= 1^{(-1)^1} \cdot 2^{(-1)^2}$

$= 1 \cdot 2^1$

$\boxed{S_2 = 2}$



$S_3 = \prod_{i=1}^{3} i^{(-1)^i}$

$= 1^{(-1)^1} \cdot 2^{(-1)^2} \cdot 3^{(-1)^3}$

$= 1 \cdot 2 \cdot 3^{-1}$

$= 1 \cdot 2 \cdot \frac{1}{3}$

$\boxed{S_3 = \frac{2}{3}}$

$S_4 = \prod_{i=1}^{4} i^{(-1)^i}$

$= 1^{(-1)^1} \cdot 2^{(-1)^2} \cdot 3^{(-1)^3} \cdot 4^{(-1)^4}$

$= 1 \cdot 2 \cdot \frac{1}{3} \cdot 4$

$\boxed{S_4 = \frac{8}{3}}$

2.66...

1, 2, 0.$\overline{66}$, 2.$\overline{66}$

strictly increasing means that $S_i < S_{i+1}$ for $1 \leq i \leq n$ in our case

$S_2 \not< S_3$

$2 \not< \frac{2}{3} \rightarrow$ no, not increasing

strictly decreasing: $S_i > S_{i+1}$

$S_1 \not> S_2$

$1 \not> 2 \rightarrow$ no, not decreasing

nonincreasing: $S_i \geq S_{i+1}$

$S_1 \not\geq S_2$

$1 \not\geq 2 \rightarrow$ not nonincreasing

nondecreasing: $S_i \leq S_{i+1}$

$S_2 \not\leq S_3$

$2 \not\leq \frac{2}{3} \rightarrow$ not nondecreasing

[redacted]

function is subset of $X \times X$ satisfying for all $x \in X$, there's one unique x such that $f(x) = y$

4. (3 points) For this problem recall that a function is a particular type of relation. Let X be a set of n elements ($n \geq 1$). Describe all functions $f : X \to X$ which are also equivalence relations. For each such function, determine the number of equivalence classes.

reflexive
symmetric
transitive

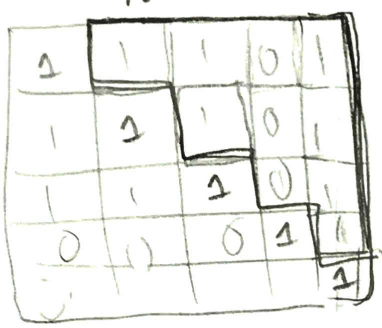
2

- X set of n elements
$n \geq 1$
X has more than one element

$f: X \to X$

n

n



To be an equivalence class,
- Must be reflexive
aka the diagonal MUST consist of all 1s
(diagonal of n elements. diagram to left is ex. when n=5)

- Must be symmetric

i	j
0	0
1	1

$i \neq j$

} 2 options
~~1 does~~

$2^{\frac{n^2-n}{2}}$

# of element's we're dealing with = $\frac{n^2-n}{2}$

- Must be transitive
if xRa
aRb then xRb

(x,a)	(a,b)	(x,b)
1	1	1
0	0	0
1	0	1/2
0	1	1/2 options

more restricting than sym. cond...

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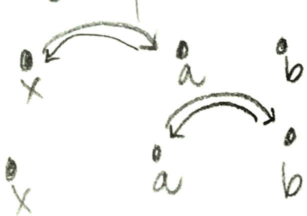
equivalence relation
- Reflexive (diagonal set)
- symmetric
- transitive
But since this is a FUNCTION
x can only map to one value (which is x)
y = x is the function
⟹ n equivalence classes



satisfies trans.

# of equivalence classes??
$[a] = \{x \in X \mid xRa\}$

$\frac{n}{\text{\# of elements in each equiv set}}$ = # of equiv. classes

⟹ cont'd.

[redacted]

5. (2 points) Two dice are rolled, one blue and one red. How many outcomes have either the blue die 3 or an even sum or both?

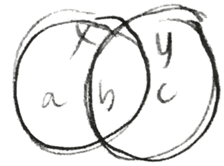
inclusion - exclusion principle

blue red

set x = outcomes blue die 3 → $\underline{3}\ \underline{1\text{-}6}$ → 6 outcomes

set y = outcomes even sum → even + even (3)(3) / odd + odd (3)(3) 9+9 = 18 outcomes

set x∩y = both conditions met (the overlap) (answer) → $\underline{3}\ \underline{\text{odd\#}}$ (3) → 3 outcomes



a+b + bc − b
a + b + c ✓

$|x \cup y| = |x| + |y| - |x \cap y|$

$|x \cup y| = 6 + 18 - 3$

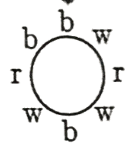
$= 24 - 3$

$= \boxed{21 \text{ outcomes}}$

inclusion exclusion principle

- dont double count

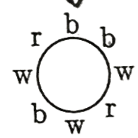
6. (3 points) There are 2 red balls, 3 white balls, and 3 blue balls. It is understood that balls of the same color are indistinguishable. The balls are to be arranged in a circle, for example like this:



bwrwbwrb

bbwrwbwr

(a) In how many ways can they be arranged if we don't distinguish between arrangements obtained from each other by rotation? I.e. we don't distinguish for example between the arrangement above and the following one:



$$\boxed{\frac{8!}{2!\,3!\,3!} \cdot \frac{1}{8}}$$

red white blue

doesnt matter where you start
- order all of them
- divide by 8 to account that rotation doesnt matter

(b) What if we add the condition that the red balls are to be placed next to each other?

$\underbrace{RR}$ WWW BBB

count as 1 "unit" since they cannot be separated

$\Rightarrow$ so now I have 7 total balls

$$\boxed{\frac{1}{7} \cdot \frac{7!}{3!\,3!}}$$

$7!$ ← total orderings

$3!\,3!$ ← to account that 3 white are indistinguishable and 3 blue indistinguishable

account that rotation indistinguishable