Math 61
Fall 2017
12/9/17
Time Limit: 180 Minutes

[redacted]

Day \ T.A.	Riley	Paul	Kevin
Tuesday	1A	1C	1E
Thursday	1B	1D	1F

This exam contains 7 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter your name and SID number on the top of this page, circle your section, and put your initials on the top of every page, in case the pages become separated. Also, have your photo ID on the desk in front of you during the exam.

Calculators or computers of any kind are not allowed. You are not allowed to consult any other materials of any kind, including books, notes and your neighbors. You may use the back of this sheet for your notes ("scratch paper"). If you need additional paper, let the proctors know.

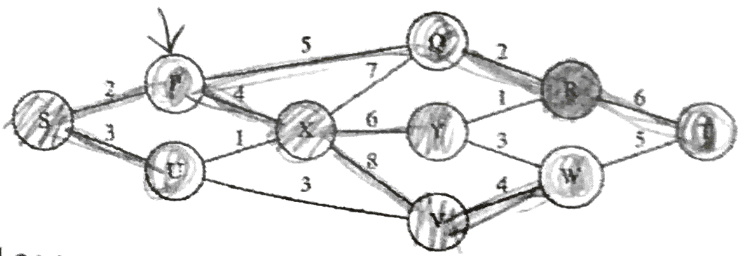
You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a result from class, discussion, or homework you must indicate this** and explain why the result may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Of course, if you have a question about a particular problem, please raise your hand and one of the proctors will come and talk to you.

Problem	Points	Score	
1	5	5	
2	5	4	+1 14
3	5	5	
4	5	5	
5	5	3	9
6	5	~~2~~	1+2
Total:	30	23	+1+2

1. (5 points) Use the breadth-first algorithm to construct a spanning tree for the following graph. (The ordering of the vertices is alphabetic.) Is the tree you obtain of minimal weight? (Here, the *weight* of a spanning tree is the sum of the weights of all its edges.)



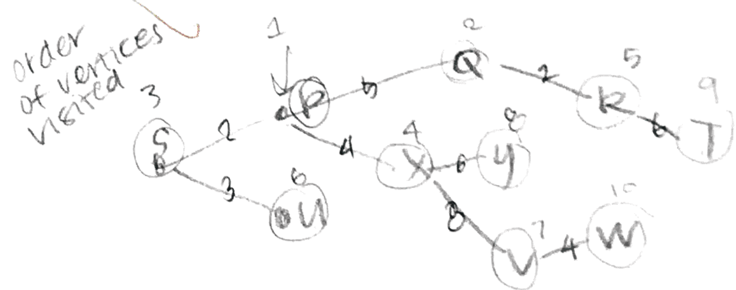
The alphabet order:

P Q R S T U V W X Y

start @ P – add (P,Q) → Then look at vertex Q add (Q,R)
(P, S)
(P, X)

Then look at vertex S → vertex X add → vertex R
add (S,U) (X,V) add (R,T)
(X,Y)

→ vertex V
add
(V,W)

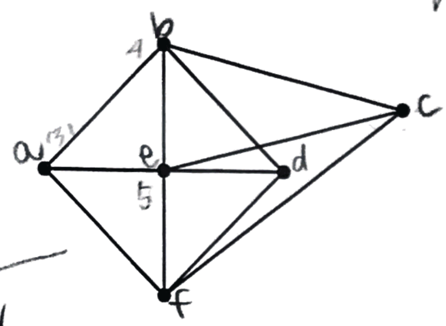


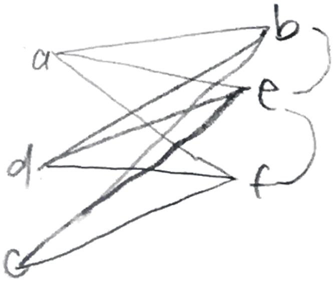
Tree weight = 2+3 + 4 + 5 + 6 + 8 + 2 + 4 + 6
5 + 9 + 14 + 6 + 6
14 + 20 + 6
34 + 6
40

not minimal weight b/c if you were to just replace edge (P,X) in our tree w/ (U,X) the total weight would decrease by 3 indicating we dont have a MST

2. (5 points) Show that the following graph is not planar.

homeomorphic to $K_{3,3}$ or K_5

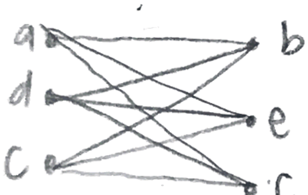




← redrew graph

we didnt even need to do series reduction. graph contains subgraph isomorphic to $K_{3,3}$

This graph is homeomorphic to $K_{3,3}$ & thm. that any graph homeomorphic to $K_{3,3}$ or K_5 isnt planar

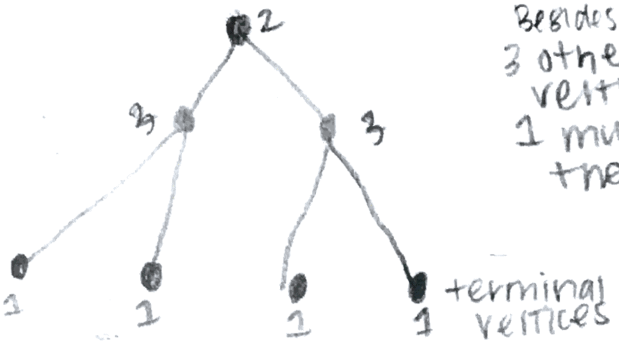


this is $K_{3,3}$

the complete bipartite graph w/ vertex set 1 consisting of vertices {a, d, c} and vertex set 2 consisting of {b, e, f}

+1

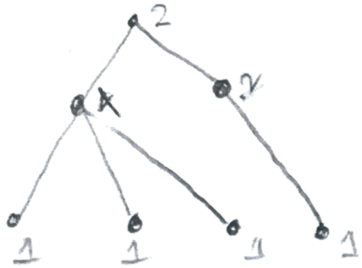
3. (5 points) G is a rooted tree with ~~9~~ 7 vertices, of which ~~5~~ 4 are leaves (=terminal vertices) (has no children). What are the possible degrees of the 9 vertices? (In particular, your answer should say why there are no other possibilities.) Draw an example of each.



Besides term. vertices, 3 other vertices, 1 must be the root

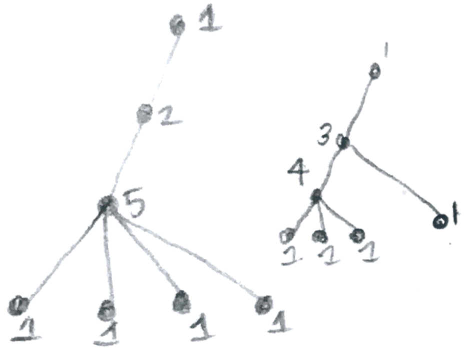
7 vertices in a tree means there are 6 edges. 4 edges must be used as the edge connecting terminal vertices
- we have 2 edges remaining to connect root & 2 other vertices

But b/c we only have 4 terminal vertices, the 2 internal vertices must have deg. at least 2. & root has at least 1 deg.



1 + 3 + 4
1 + 2 + 5
2 + 3 + 3 same → 3 + 2 + 3
2 + 4 + 2

4 possible outcomes for degrees of root & internal only
that's why we have 4 possible degree sequences
ex. trees on side



(1, 1, 1, 1, 7/1, 7/2, 7/2) total degrees 12

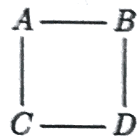
assuming you don't care if root is 2, internal vertices are both 3 vs. root is 3 internal vertices degree 2 and 3 counted that as same deg. sequence

$\overset{x}{7/1} + \overset{y}{7/2} + \overset{z}{7/2} = 8$

no other possibilities b/c
if root had degree 0, G wouldn't be a tree (not connected)
if internal vertices only had degree 1 then it would be a terminal vertex which isn't what we want (we already separated our 4 terminal vertices)

5

4. (5 points) A frog jumps on the vertices of a square, every time jumping to one of the two closest vertices. So, for example, from A it jumps to either B or C.



In how many different ways can it get from A to A in n jumps?

Hint: it might be useful to consider the number of ways to get from A to B, C, and D in n jumps as well.

path length	1	2	3	4	5	6
# paths length (n)	0	2	0	8	0	32

adjacency matrix

$A=$

	a	b	c	d
a	0	1	1	0
b	1	0	0	1
c	1	0	0	1
d	0	1	1	0

0	1	1	0
1	0	0	1
1	0	0	1
0	1	1	0

$a \to a \to a$ $a \to b + b \to a$ $a \to c + c \to a$

$A^2 =$

2	0	0	2
0	2	2	0
0	2	2	0
2	0	0	2

$A^3 =$

0	4	4	0
4	0	0	4
4	0	0	4
0	4	4	0

0000

$A^4 =$ $A^5 = 0$

S_n = # ways to get from A to A in n jumps

$A^{n-1} \cdot A^1$ A^n

$$S_n = \begin{cases} 2^{n-1} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

$A^4 =$

8	0	0	8
0	8	8	0
0	8	8	0
8	0	0	8

$A5 =$ 0 16 16 0

$A6 =$ 32 0 0 32

5. (5 points) You don't need to simplify the formulas in this problem.

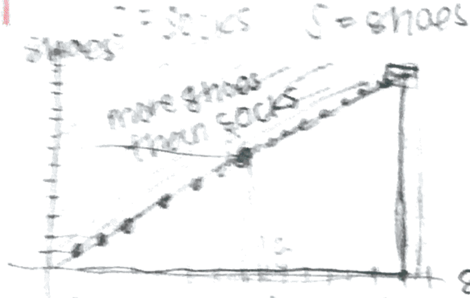
(a) Suppose you want to buy 12 items from an ample supply of glazed, chocolate, and powdered donuts. How many selections are possible?

(b) A 9-digit telephone number $d_1d_2d_3d_4d_5d_6d_7d_8d_9$ is called *memorable* if the sequence $d_1d_2d_3d_4$ is exactly the same as one of the sequences $d_5d_6d_7d_8$ or $d_6d_7d_8d_9$ (or both). Assume that each d_i can be any of the ten decimal digits 0,1,...,9. What is the number of memorable telephone numbers?

(c) In how many different orders can a spider put on its socks and shoes? (Of course, a spider has eight legs, and socks are put on before shoes.)

(a) indistinguishable
stars & bars method
☆ ☆ ☆ ☆ ☆ ☆ ☆ ☆ ☆ ☆ ☆ ☆ ☆ ☆
glazed chocolate powdered
C(14, 2)

(c) spider - 8 legs
socks before shoes
= socks S = shoes



dont want to go above diagonal
is this a catalan # thing ...?

16 total items to order
(8 socks, 8 shoes)

wouldnt this just be all possibilities divided by 2? half of them will be below diag. half will be above

$\frac{16!}{2}$

(b) 1 2 3 4 same as 5 6 7 8
or same as 6 7 8 9

Inclusion Exclusion

all possible phone #s − phone numbers where 1234 match 5678 − phone #s where 1234 match 6789

+ phone #s where 1234 match 5678 & 6789

all possible 10^9 − #s 1234 match 5678 10^5 − #s 1234 match 6789 10^5

+ phone #s where 1234 match 5678 & 6789
10^4

$10^9 - 10^5 - 10^5 + 10^4$

6. (5 points) Prove that from any set of 1000 positive integers, one can choose either one number which is divisible by 1000, or several numbers whose sum is divisible by 1000. unm... ^^

Hint: Denote the numbers by $x_1, x_2, \ldots, x_{1000}$. *Consider the sums*

$$x_1, \quad x_1 + x_2, \quad \ldots, \quad x_1 + x_2 + \ldots + x_{1000}.$$

pigeon hole

If one of them is divisible by 1000, then … Otherwise …

$x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad .. \quad x_{1000}$

you pick 1000 #s
Divisibility by 1000
0%1000 1%1000 ...... 999%1000
we have 1000 buckets

If no sum of several #s are div. by 1000 that means you chose 1 # to put into each of the 1000 pigeon holes which means 1 # falls into pigeon hole 0%1000 so it's divisible by 1000

If one # isnt divisible by 1000 then no # falls into pigeon hole 0%1000 so only have 1%1000, ~ ~ 999%1000 as pigeonholes
999 buckets
1000 #s

1 bucket has at least 2 #s by PHP

Theres always going to be some way to add several #s where their sum is divisible by 1000

$x_1, x_1+x_2, x_1+x_2+x_3, \ldots, x_1+x_2+\ldots x_{1000}$
$x_2+x_3, x_2+x_3+x_4, \ldots x_2+x_3+\ldots x_{1000}$
$x_3+x_4,$

you have a lot of possible sums... More than 1000 possible sums and we want the sum to be 0%1000 if you take all possible sums w/ pigeon holes

0%1000, 1%1000 ...... 999%1000

lets say each bucket 1%1000 to 999%1000 each hold 999 sums then and 0%1000 holds 0 sums

adding another sum would mean that the sum of all #s in the bucket w/ 1000 sums would be 0%1000 & we have more possible sums than (999)(999) so

therefore we can find sum of several #s whose sum divisible by 1000

+2