Math 61

Midterm 2

Fall 2016

Nan

SID

Sect

There are 5 questions. Write clearly, show all of your work, and justify all of your answers. No calculators are allowed.

1	5
2	10
3	11
4	15
5	14
Total	55

1. (a) (5 pts) Show that $C(i,k) = C(i+1,k+1) - C(i,k+1)$ for $i > k$.

$$\binom{i+1}{k+1} - \binom{i}{k+1}$$

$$= \frac{(i+1)!}{(k+1)!\,((i+\cancel{1})-(k+\cancel{1}))!} - \frac{i!}{(k+1)!\,(i-(k+1))!}$$

$$= \frac{i!}{k!\,(i-k)!}\cdot\frac{i+1}{k+1} - \frac{i!}{k!\,(i-k)!}\cdot\frac{i-k}{k+1}$$

$$= \frac{i!}{k!\,(i-k)!}\left(\frac{\cancel{i}+1}{k+1} - \frac{\cancel{i}-k}{k+1}\right)$$

$$= \binom{i}{k}\left(\cancel{\frac{1+k}{k+1}}\right)$$

$$\overset{\checkmark}{=} \binom{i}{k}$$

5

(b) (5 pts) Show that

$$C(n+1,k+1) = \sum_{i=k}^{n} C(i,k).$$

$$\binom{n+1}{k+1}$$

$$= \frac{(n+1)!}{(k+1)!\,((n+\cancel{1})-(k+\cancel{1}))!}$$

$$= \frac{n!}{k!\,(n-k)!}\cdot\frac{n+1}{k+1}$$

$$= \binom{n}{k}\frac{n+1}{k+1}$$

~~By the binomial theorem, the right side is equal to 2^n, which is the number of possible subsets of a set of size k. But the left side also counts the number of possible subsets of a set of size k, so the two sides are equal.~~

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$$\sum_{i=k}^{n}\binom{i}{k} = \sum_{i=k}^{n}\left[\binom{i+1}{k+1} - \binom{i}{k+1}\right]$$

0

$$= \left[\cancel{\binom{k+1}{k+1}} - \cancel{\binom{k}{k+1}}^{\,0}\right] + \left[\cancel{\binom{k+2}{k+1}} - \cancel{\binom{k+1}{k+1}}\right] + \left[\cancel{\binom{k+3}{k+1}} - \cancel{\binom{k+2}{k+1}}\right] + \cdots + \left[\binom{n+1}{k+1} - \cancel{\binom{n}{k+1}}\right]$$

$$\overset{\checkmark}{=} \binom{n+1}{k+1}$$

10

2. (a) (5 pts) Find the general solution for the recurrence $a_n = -4a_{n-1} - 4a_{n-2}$.

· guess: $a_n = \lambda^n \rightarrow \lambda^n = -4\lambda^{n-1} - 4\lambda^{n-2}$

· char poly: $0 = \lambda^2 + 4\lambda + 4 = (\lambda+2)^2 \rightarrow \lambda = -2$

· gen soln: $a_n = c_1(-2)^n + c_2 n(-2)^n$

✓

(b) (5 pts) Find the solution to the recurrence $a_n = -4a_{n-1} - 4a_{n-2}$ for $n \geq 2$ with initial conditions $a_0 = 3$ and $a_1 = 0$.

· init cond: $\begin{cases} 3 = c_1(-2)^0 \cancel{+ c_2 \cdot 0 \cdot (-2)^0} \\ 0 = c_1(-2)^1 + c_2 \cdot 1 \cdot (-2)^1 \end{cases} \rightarrow \begin{cases} 3 = c_1 \\ 0 = -2c_1 - 2c_2 \end{cases}$

$0 = c_1 + c_2$

$0 = 3 + c_2$

$c_2 = -3$

$\therefore a_n = 3(-2)^n - 3n(-2)^n$

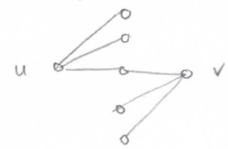
✓

3. (a) (10 pts) Suppose $G = (V, E)$ is a simple graph with n vertices. Prove that if $\deg(v) \geq \frac{n-1}{2}$ for every vertex $v \in V$, then G is connected.

10

Assume $n > 1$ (for $n \leq 1$, the graph is trivially "connected"). Let u, v be any two vertices in V. There are two cases:

- u and v are directly connected
- u and v are not directly connected, but are each connected to at least $\frac{n-1}{2}$ of the $n-2$ remaining vertices that are not u or v. But since $\deg(u) + \deg(v) \geq \frac{n-1}{2} + \frac{n-1}{2} = n-1 > n-2$, there must be at least one vertex connected to both u and v. Then u and v are connected.



$\therefore$ Any two vertices u, v in V are connected, so G is connected.

(b) (5 pts) Suppose $G = (V, E)$ is a simple graph with n vertices where $n > 1$. Prove that there must be two different vertices of G that have the same degree.

1

ok

- Base case: For $n = 2$, the two vertices must have the same degree, since all edges are between the two vertices.
- Inductive step: If the proposition is true for $n = k$, then it must be true for $n = k+1$. When n goes from even to odd . . .

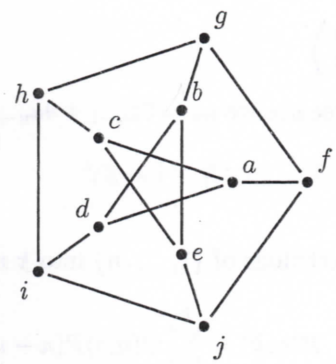
Since G has n vertices, its vertices can have degree $0, 1, \ldots, n-1$ for a total of n possible values. But vertices of degree 0 and degree $n-1$ cannot coexist in a single graph, since a vertex of degree $n-1$ must be connected to every other vertex; then there can only be $n-1$ distinct values for the degree of a vertex in G. By the pigeonhole principle, there must be two different vertices of G that have the same degree.

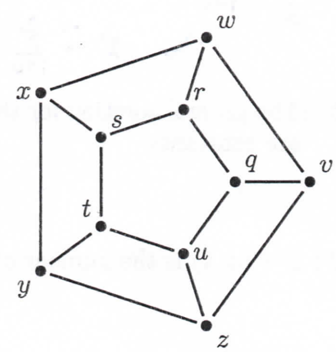
4. (a) (5 pts) State the definition of when two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic.

$G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic iff $\exists$ a bijection $f: V_1 \to V_2$ s.t.
$[\, \forall u, v \in V_1 \;\; (u,v) \in E_1$ iff $(f(u), f(v)) \in E_2 \,]$

(b) (5 pts) Are the following two graphs isomorphic? If so, give an isomorphism using the table below. If not, explain why.

$a \mapsto$
$b \mapsto$
$c \mapsto$
$d \mapsto$
$e \mapsto$
$f \mapsto$
$g \mapsto$
$h \mapsto$
$i \mapsto$
$j \mapsto$

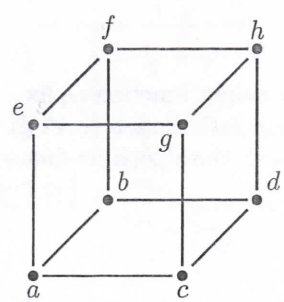


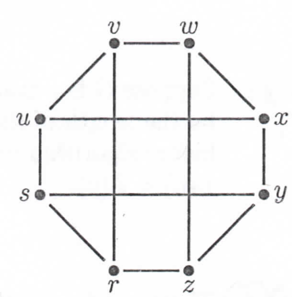


No. The right graph contains 4-cycles but the left graph does not.

(c) (5 pts) Are the following two graphs isomorphic? If so, give an isomorphism using the table below. If not, explain why.

$a \mapsto r$
$b \mapsto v$
$c \mapsto z$
$d \mapsto w$
$e \mapsto s$
$f \mapsto u$
$g \mapsto y$
$h \mapsto x$





15

5. (20 pts) Circle whether the following are True or False. **You do not need to justify these answers.**

T / F: There is a simple graph with 6 vertices whose vertices have degree 0, 1, 2, 3, 4, and 4.

T / F: The graph $K_{3,3}$ has exactly $3^2 2^4 = 144$ simple cycles.

T / F: $4^n = \sum_{i=0}^{n} 2^n C(n,i)$ for every integer $n \geq 0$.

3^n

$2^n \stackrel{?}{=} \sum_{i=0}^{n} \binom{n}{i}$

T / F: The general solution for the recurrence $a_n = 6a_{n-1} + 9a_{n-\cancel{1}\,2} + 2^n$ is $a_n = b_1 3^n + b_2 2^n$ where b_1, b_2 are constants.

$0 = \lambda^2 - 6\lambda - 9 = (\lambda - 3)^2$

T / F: If $P(n,k)$ is the number of partitions of $\{1, \ldots, n\}$ into k many pieces, then for $n, k > 1$,

$$P(n,k) = \sum_{i=1}^{k} C(n,i) P(n-i, k-1).$$

$P(n+1, k+1) = \sum_{i=k}^{n} \binom{n}{i} P(i,k)$

T / F: If G is a graph with n vertices, then any path of length n in G must include some vertex at least twice.

T / F: Suppose A is the adjacency matrix of a graph G with n vertices. If G is connected, then every entry of the matrix A^n is nonzero.

T / F: Suppose G is a graph with weight function w, fix a vertex a, and for every vertex v in G, let $L(v)$ be the length of the shortest path from a to V. If we use Dijkstra's algorithm to find $L(v)$, then before algorithm returns $L(v)$, the algorithm first correctly finds $L(u)$ for every vertex u such that $L(u) < L(v)$.

T / F: There are $(n-1)!$ different isomorphisms from K_n to K_n.

$n!$

T / F: If G is a simple graph that has an Euler cycle, and G' is a subgraph obtained from G by removing only a single edge and removing no vertices, then G' cannot have an Euler cycle.

14