Math 61

Midterm 1

Fall 2016

Nam

SID:

Secti

There are 5 questions. Write clearly, show all of your work, and justify all of your answers. No calculators are allowed.

1	15
2	10
3	6
4	15
5	16
Total	62

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1. (15 pts)

(a) (5 pts) Write the definition of what it means for a relation R on a set X to be an equivalence relation.

a relation R on X is an equivalence relation iff R is reflexive, symmetric, and transitive.

(b) (6 pts) Let E be the relation on the set of all positive real numbers $\mathbb{R}^+$ where $x\ E\ y$ if $x/y = 2^n$ for some integer $n \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. For example, $(2/3)\ E\ (8/3)$, since $\frac{2/3}{8/3} = \frac{1}{4} = 2^{-2}$. Show that E is an equivalence relation.

reflexive: yes. $\forall x \in \mathbb{R}^+, \frac{x}{x} = 1 = 2^0$ (i.e., $n = 0$) $\Rightarrow x E x$.

symmetric: yes. $\forall x, y \in \mathbb{R}^+, x E y \Rightarrow \frac{x}{y} = 2^n, n \in \mathbb{Z} \Rightarrow \exists m \in \mathbb{Z}, m = -n,$ s.t. $\frac{y}{x} = 2^m \Rightarrow y E x$

transitive: yes, $\forall x, y, z \in \mathbb{R}^+, x E y \wedge y E z \Rightarrow \frac{x}{y} = 2^n, \frac{y}{z} = 2^m; n, m \in \mathbb{Z}$ $\Rightarrow \exists p \in \mathbb{Z}, p = n + m,$ s.t. $\frac{x}{z} = \frac{x}{y}\frac{y}{z} = 2^n 2^m = 2^{n+m} = 2^p \Rightarrow x E z$

E is reflexive, symmetric, and transitive $\Rightarrow$ E is an equivalence relation.

(c) (4 pts) What is the equivalence class of 1 with respect to the relation E?

$[1] = \{x \mid x = 2^n, n \in \mathbb{Z}\}$

2. (15 pts)

Prove the following by induction for every $n \geq 1$. For all finite sets X and Y with $|X| = |Y| = n$, if f is a function from X to Y that is one-to-one, then f is onto.

(a) (3 pts) State and prove the base case.

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The proposition is true for the base case $n = 1$:

The one element in X must be mapped to the one element in Y.
Then $\forall y \in Y\ \exists$ at least one $x \in X$ s.t. $f(x) = y$ and f is onto.

(b) (9 pts) Prove the inductive step.

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The proposition is true for $n+1$ if it is true for n:

When we increase from n to $n+1$, we add an element to X and an element to Y. In order for f to remain a function ($\forall x \in X\ \exists$ exactly one $y \in Y$ s.t. $f(x) = y$) and one-to-one ($\forall x, y \in X$, $f(x) = f(y) \Rightarrow x = y$), the new element in X must be mapped to the new element in Y. With the new element in Y mapped, we have $\forall y \in Y\ \exists$ at least one $x \in X$ s.t. $f(x) = y$ and f is onto.

$\therefore$ The proposition is true for $n \geq 1$.

You need to show that for all 1-1 functions f from X to Y with $|X| = |Y| = n+1$, f is onto.

Why is every such function constructed the way you describe?

(c) (3 pts) Either prove the following or give a counterexample: For every set X, if f is a one-to-one function from X to X, then f is onto.

0

If f is a one-to-one function from X to X, then f is also a one-to-one function from X to Y $(= X)$, $|X| = |Y| = n$. Thus the preceding proof applies and f is onto.

False. Suppose $f: \mathbb{N}_0 \to \mathbb{N}_0$ and $f(x) = x+1$. Then f is a one-to-one function from $\mathbb{N}_0$ to $\mathbb{N}_0$ but f is not onto since 0 is not in the image.

3. (10 pts) **In the following problem, for full credit, state any counting rules or principles that you use. You do not need to simplify your answers (for example, they may contain factorials). However, your final answer may not include the functions $P(n, r)$ or $C(n, r)$.**

(a) (5 pts) How many solutions are there to $x_1 + x_2 + \ldots + x_9 = 20$, where $x_1, x_2, \ldots, x_9$ are integers greater than or equal to 0?

We need to separate the sum of 20 into 9 integers greater than or equal to 0. Use stars-and-bars with 20 stars and $9-1=8$ bars:

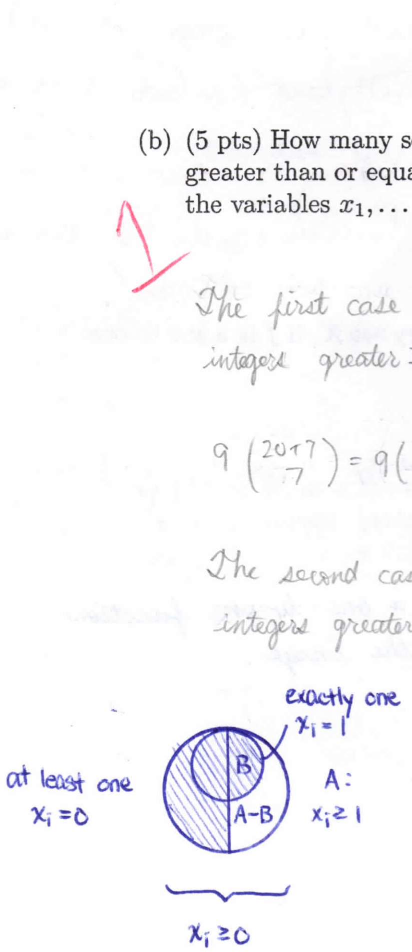
$$\binom{20+8}{8} = \binom{28}{8} = \frac{28!}{8!\,20!}$$

(b) (5 pts) How many solutions are there to $x_1 + x_2 + \ldots + x_9 = 20$, where $x_1, x_2, \ldots, x_9$ are integers greater than or equal to 0, and at least one of the variables $x_1, \ldots, x_9$ is equal to 0 or exactly one of the variables $x_1, \ldots, x_9$ is equal to 1?

The first case is equivalent to separating the sum of 20 into 8 integers greater than or equal to 0, but the required 0 can go anywhere:

$$9\binom{20+7}{7} = 9\binom{27}{7} = 9 \cdot \frac{27!}{7!\,20!}$$

The second case is equivalent to separating a sum of 19 into 8 integers greater than or equal to 0, but the required 1 can go anywhere:



Consider the Euler diagram to the left. The answer is the number of solutions in the shaded area.

- By stars and bars, the total number of solutions where $x_i \geq 0$ is $\binom{20+8}{8} = \binom{28}{8}$.
- By stars and bars, if we pre-allocate one star to each x_i, then $|A| = \binom{(20-9)+8}{8} = \binom{19}{8}$.
- By stars and bars, if we allocate exactly one star to one of the 9 x_i and pre-allocate two stars to each of the 8 remaining x_i, then $|B| = 9 \cdot \binom{(20-1-8\cdot 2)+7}{7} = 9 \cdot \binom{10}{7}$.
- Then the number of solutions in the shaded area is $\binom{28}{8} - \left[\binom{19}{8} - 9 \cdot \binom{10}{7}\right] = \frac{28!}{8!\,20!} - \left[\frac{19!}{8!\,11!} - 9\,\frac{10!}{7!\,3!}\right]$.

4. (15 pts) Consider a standard deck of 52 playing cards, where each card is one of four different suits $\diamondsuit, \heartsuit, \clubsuit, \spadesuit$ and one of 13 different ranks: A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, and all combinations of suits and ranks are possible.

A hand means 5 different cards where order does not matter.

For full credit, state any counting rules or principles that you use. You do not need to simplify your answers (for example, they may contain factorials). However, your final answer may not include the functions $P(n,r)$ or $C(n,r)$.

(a) (5 pts) How many hands contain only cards whose rank is J or Q or K?

Choose 5 cards from the $4 \cdot 3 = 12$ face cards:

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$$\binom{12}{5} = \frac{12!}{5!7!}$$

(b) (5 pts) How many hands are "three of a kind" (contain three cards of one rank, and the remaining cards have two other ranks)?

Choose 1 from 13 ranks, then choose 3 from 4 cards of that rank. For the other cards, choose 2 from 12 remaining ranks, then choose 1 from 4 cards of each of those 2 ranks:

5

$$\binom{13}{1}\binom{4}{3} \cdot \binom{12}{2}\binom{4}{1}^2 = 13\,\frac{4!}{3!1!} \cdot \frac{12!}{2!10!}(4)^2$$

(c) (5 pts) How many hands are "two pair" (contain two cards of the same rank, two cards of another rank, and one card of a third rank)?

Choose 2 from 13 ranks, then choose 2 from 4 cards of each of those 2 ranks. For the other card, choose 1 from 11 remaining ranks, then choose 1 from 4 cards of that rank:

5

$$\binom{13}{2}\binom{4}{2}^2 \cdot \binom{11}{1}\binom{4}{1} = \frac{13!}{2!11!}\left(\frac{4!}{2!2!}\right)^2 \cdot (11)(4)$$

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5. (20 pts) Circle whether the following are True or False. **You do not need to justify these answers.**

T / F: If A and B are finite sets and $|A \cup B| = |B|$, then $A \subseteq B$.

T / F: If X is a finite set with $|X| = k$, then there are k^n many strings over X of length n.

T / F: If f is a bijection from X to Y, then f^{-1} is a bijection from Y to X.

T / F: There are $\frac{(n+m)!}{n!m!}$ ways to divide n many identical balls into m many distinct boxes.

$\binom{n+m}{m}$ should be $\binom{n+m-1}{m-1}$

T / F: Consider the set S of strings of length 10 containing exactly four a's, three b's, and three c's. There are more strings in S ending with a than there are strings in S ending with b.

$\frac{9!}{3!3!3!} > \frac{9!}{4!2!3!}$

T / F: Suppose f is a function from X to Y. Then f is one-to-one if for every $x \in X$ there is a unique $y \in Y$ so that $f(x) = y$.

T / F: Suppose R is a relation on a set X and R is transitive and symmetric. Then for all $x, y \in X$, if $(x, y) \in R$, then $(x, x) \in R$.

T / F: The relation $\{(1,2), (3,1), (2,3), (1,1), (2,2), (3,3)\}$ on the set $\{1,2,3\}$ is transitive.

$1 R 2$ and $2 R 3$ but $1 \not R 3$.

T / F: The number of onto functions from $\{1,2,3,4,5\}$ to $\{1,2\}$ is equal to $\frac{5!}{3!2!}2^3$.

$2^5 - 2$ $\quad$ $\binom{5}{2} 2^3$

T / F: If $n \geq 3$, then there are $(n-1)!/2$ ways to seat n different people around a circular table where two seatings are considered identical if each person has the same (unordered) set of two neighbors.