a b

a ab λ

b ba

1. (2 points each) **Multiple choice:** Circle the right answer. You do NOT need to justify your answers.

10 11

If α is a string of length two, what is the number of substrings of α?

1 10 1 11

(A) 2; (B) 3; (C) 4; (D) 5; (E) It depends on the string.

0 01 λ

λ

✓

In how many ways can we distribute 12 (identical) copies of the same book to 5 people: Alice, Bob, Casey, David and Emily?

(A) 12^5; (B) 5^{12}; (C) $C(12,5)$; (D) $C(16,4)$; (E) $C(16,11)$.

12+5−1, 5−1

C(16,4)

✓

In how many ways can we distribute 12 distinct books to 5 people: Alice, Bob, Casey, David and Emily?

5^{12}

(A) 12^5; (B) 5^{12}; (C) $C(12,5)$; (D) $C(16,4)$; (E) $C(16,11)$.

✓

Let $X = \{1, 2, 3, \dots\}$ be the set of all natural numbers. If $f, g : X \to X$ are two functions such that $f \circ g$ is bijective, then:

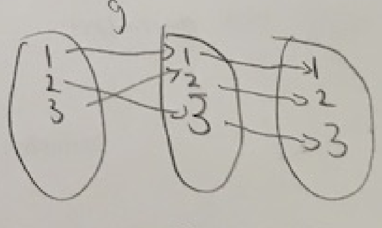
(A) f has to be bijective, but g does not have to be bijective;

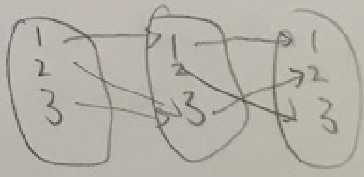
(B) g has to be bijective, but f does not have to be bijective;

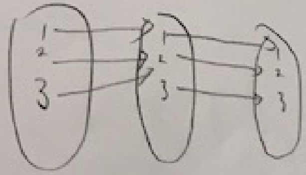
(C) Both f and g have to be bijective;

(D) Neither f nor g have to be bijective.

✗







2. Write down the answer to each question. *You do NOT need to justify your answers.* Also, you do not need to simplify expressions such as 2^6, $6!$, $C(6,3)$, etc.

(a) Consider the set $X = \{1, 2, \ldots, 7\}$.

(2 points) How many of the relations on X are NOT reflexive?

$2^{n^2} - 2^{n^2-n}$

$\boxed{2^{49} - 2^{42}}$

Reflexive → 1's along diagonal

(2 points) How many of the relations on X are symmetric?

2^{n^2-n} relations that are ~~diagonal~~ reflexive (diagonal must be 1)

$2^{\frac{n^2-n}{2}} \cdot 2^n$

$2^{21} \cdot 2^7 = \boxed{2^{28}}$

(2 points) How many of the relations on X are symmetric OR reflexive (i.e. either symmetric, or reflexive, or both)?

both symmetric and reflexive

$2^{\frac{n^2-n}{2}} \cdot 2^n + \underbrace{2^{n^2-n}}_{\text{reflexive}} - \underbrace{2^{\frac{n^2-n}{2}}}$

inclusion exclusion

$\boxed{2^{28} + 2^{42} - 2^{21}}$

(2 points) How many of the relations on X are symmetric but NOT reflexive?

$\cancel{2^7}$ $2^7 - 2$ possible for not reflexive $2^{\frac{n^2-n}{2}}$

→ all 0's along the diagonal

$\boxed{(2^7-2)2^{21}}$

$\cancel{\boxed{2^{21}}}$

(2 points) How many of the relations on X are both symmetric AND antisymmetric?

along diagonal.
n possible
entry's.

2^7 $\boxed{2^7}$

Symmetric and
anti symmetric
~~at~~ can only
happen with something that is only reflexive
~~reflexiveness~~

(b) (2 points) How many distinct strings can be obtained from the string $\underbrace{AAAA}_{4}B\underbrace{CC}_{3}\underbrace{DD}_{2}$ by permuting (re-ordering) its letters?

9

$$9C4 \cdot 5C1 \cdot 4C2 \cdot 2C2$$

$$\frac{9! \; \cancel{5!} \; \cancel{4!} \; \cancel{2!}}{4!\,\cancel{5!}\;1!\,\cancel{4!}\;2!\,\cancel{2!}\;0!\,2!}$$

$$\boxed{\frac{9!}{4!2!2!}}$$

3. Let $\mathbb{Z}$ be the set of all integers, and $\mathcal{P}(\mathbb{Z})$ the power set of $\mathbb{Z}$ (consisting of all subsets of $\mathbb{Z}$). Consider the following relation R on $\mathcal{P}(\mathbb{Z})$:

$$(A, B) \in R \iff A \cap B = \emptyset.$$

Answer the following questions, fully justifying your answers. (If an answer is YES, explain why. If an answer is NO, give a counterexample.)

(a) (2 points) Is R reflexive?

No because, for example lets take the set $\{1\}$. $\{1\} \cap \{1\} = \{1\}$ because both sets have 1. Since this means $\{1\} \cap \{1\}$ does not give an empty set, R is not reflexive.

(b) (2 points) Is R symmetric?

Yes, because $A \cap B = B \cap A$ for all subsets of $\mathbb{Z}$

(c) (3 points) Is R transitive?

(1, 2) (3, 4) (2, 5)

No. For example let $A \cap B$ for $A = \{1, 2\}$ and $B = \{3, 4\}$. $A \cap B = \emptyset$. Lets ~~att~~ also take $B \cap C$ for $B = \{3, 4\}$ and $C = \{2, 5\}$, $B \cap C = \emptyset$. If we take $A \cap C$, we have $\{1, 2\}$ and $\{2, 5\}$, and $A \cap C = \{2\}$. Because $A \cap C$ is not empty, R is not transitive.

(d) (3 points) Prove that $(A, B) \in R \circ R$ for all A and B.

~~Matrix:~~

~~Because R is symmetric (as proven in part b), we have $A \cap B$ and $B \cap A$ is R.~~

~~$A \cap B$~~

$A \cap B = \emptyset$ in R

That means $R \circ R = \emptyset$ .....

?

So, this implies that $A \cap B = \emptyset$, so $(A, B) \in R \circ R$.

~~10~~ 9

4. (10 points) Prove by induction on n that:

$$2^{2n} \leq (n+2)!,$$

for any integer $n \geq 1$.

Base case: $n=1$ $\quad 2^{2\cdot 1} \leq (1+2)!$

$$2^2 \leq 3!$$

$$4 \leq 6 \checkmark$$

Inductive hypothesis: Assume $S(n)$, $2^{2n} \leq (n+2)!$

$S(n+1)$ Inductive Step: We want to show $2^{2(n+1)} \leq (n+3)!$

By inductive hypothesis $\quad (n+3)! = (n+2)!(n+3) \geq 2^{2n}(n+3)$

It suffices to show:

$$2^{2n}(n+3) \geq 2^{2(n+1)}$$

$$\cancel{2^{2n}}(n+3) \geq \cancel{2^{2n}} \cdot 2^2$$

$$(n+3) \geq 4$$

Because $n \geq 1$, this must always be true. need $S(n) \Rightarrow S(n+1)$ ~~technically~~

Because $S(n+1) \Rightarrow S(n)$,

$S(n)$ is true for all $n \geq 1$.

Do not write on this page.

18	1	6	out of 8 points
	2	12	out of 12 points
16	3	7	out of 10 points
	4	9	out of 10 points
	Total	34	out of 40 points