**Name:** ______________________________

**UCLA ID Number:** ______________________________

**Section letter:** ____________

**Math 61 : Discrete Structures**
**Final Exam**
Instructor: Ciprian Manolescu

You have 180 minutes.

No books, notes or calculators are allowed.
Do not use your own scratch paper.

**1.** (2 points each) **True/False:** Circle the right answers. You do NOT need to justify your answers.

There exists a function from the empty set $\emptyset$ to the set $\{1\}$. **T**

There exists a function from the set $\{1\}$ to the empty set $\emptyset$. **F**

Any surjective function $f : \{1, 2, \ldots, 100\} \to \{1, 2, \ldots, 100\}$ is bijective. **T**

We want to select a team of 7 (unordered) players out of 60 people.
This can be done in exactly $\frac{60!}{53!}$ ways. **F**

When we expand the expression $(x - y + z)^{10}$, the coefficient of $x^2y^4z^4$ is $\frac{10!}{2!4!4!}$. **T**

Let $G$ be the graph with one vertex and no edges. Then $G$ is bipartite. **T**

There exists a graph with 9 vertices, all of degree 3. **F**

The Selection Sort algorithm sorts $n$ objects in time $\Theta(n \lg n)$. **F**

$\lg(2^n + 3^n) = \Theta(n)$. **T**

$\lg(2^n + (n!)^2) = \Theta(n \lg n)$. **T**

**2.** (5 points each) Write down the answer to each question. *You do NOT need to justify your answers.*

The number of functions $f : \{1, \dots, 10\} \to \{1, \dots, 15\}$ that are NOT injective is:

$$15^{10} - \frac{15!}{5!}$$

Determine the number of relations on the set $\{1, 2, \dots, n\}$ that are either symmetric, or reflexive, or both:

$$2^{n^2-n} + 2^{n(n+1)/2} - 2^{n(n-1)/2}$$

In how many ways can we distribute 100 identical cookies to 10 people, such that the $k$th person gets at least $k$ cookies, for all $k$? (In other words, person 1 gets at least one cookie, person 2 gets at least two cookies, ... , person 10 gets at least ten cookies.)

$$\sum_{k=1}^{10} a_k = 100 \text{ with } b_k = a_k - k \geq 0$$

means

$$\sum_{k=1}^{10} b_k = 100 - (1 + 2 + \cdots + 10) = 100 - (10 \cdot 11)/2 = 100 - 55 = 45$$

so the number is

$$C(10 + 45 - 1, 10 - 1) = C(54, 9)$$

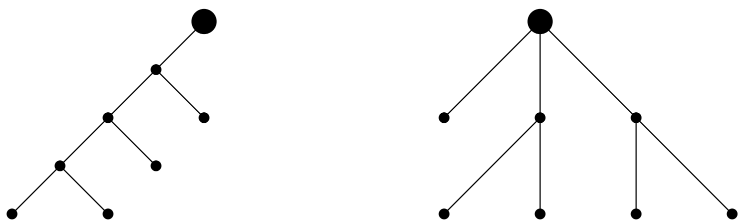
Calculate the sum:

$$C(51,0)+C(51,1)+C(51,2)+\cdots+C(51,24)+C(51,25).$$

We know that $C(51,k)=C(51,51-k)$. Therefore,

$$C(51,0)+\cdots+C(51,25)=C(51,26)+\cdots+C(51,51)=\frac{1}{2}(C(51,0)+\cdots+C(51,51))=\frac{1}{2}2^{51}=2^{50}.$$

The trees



(with the roots drawn as the bigger dots) are:

(A) isomorphic as both free trees and rooted trees;

$\boxed{(B)}$ isomorphic as free trees, but not as rooted trees;

(C) isomorphic as rooted trees, but not as free trees;

(D) not isomorphic as free trees, and not isomorphic as rooted trees.

**3.** (10 points) Let $X \subset \{1, 2, \ldots, 100\}$ be a subset consisting of 57 numbers. Show that we can find $a \in X$ such that $a + 13 \in X$. *Fully justify your answer.*

Consider the elements of $X$ and $X' = \{a + 13 \mid a \in X\}$. Together they are $57 \cdot 2 = 114$ elements from 1 to 113. By the pigeonhole principle, two are equal.

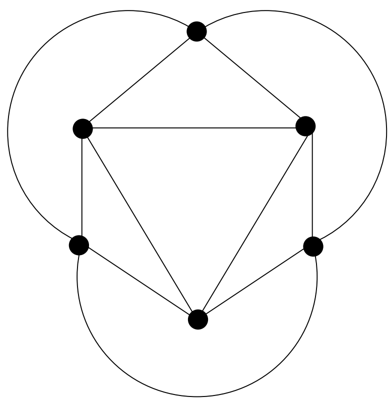
**4.** A simple, connected, planar graph $G$ has each vertex of degree 4.

(a) (6 points) Show that $G$ has at least 8 faces (including the outer face). *Fully justify your answer.*

The sum of the degrees is $4v = 2e$, so $e = 2v$. Euler's formula gives $v - e + f = 2$, so $v - 2v + f = 2$, and hence $v = f - 2$ and $e = 2(f - 2)$. Since each face has at least 3 edges, we have $3f \leq 2e = 4(f - 2)$, which implies $f \geq 8$.

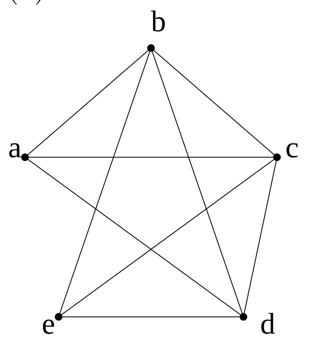
(b) (4 points) Give an example of such a graph $G$, with each vertex of degree 4 and exactly 8 faces.

Note that in order to have equality in the argument above, each face must have exactly 3 edges. Also, there must be 12 edges and 6 vertices. Here's an example:

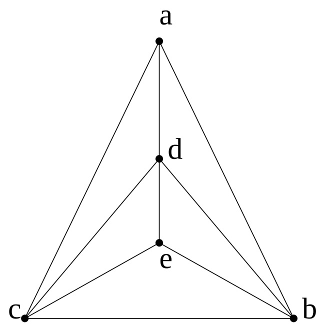


**5.** (5 points each) Determine if the following graphs are planar. *If the graph is planar, redraw it so that the edges do not cross. If the graph is not planar, explain why.*

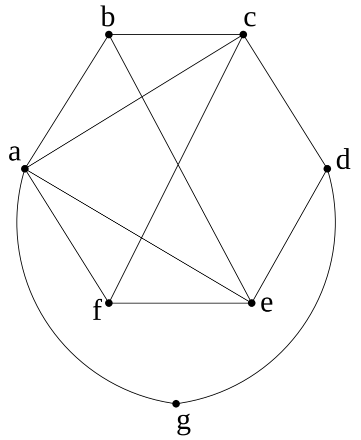
(a)



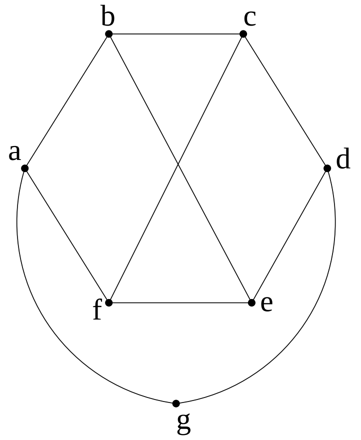

This is planar, it can be redrawn as:



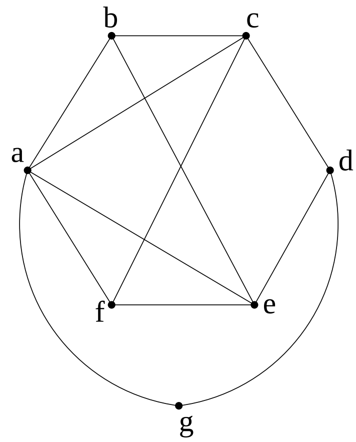
(b)




This is not planar, as it contains a graph homeomorphic to $K_{3,3}$, with the bipartition being $a, c, e$ versus $b, d, f$:



**6.** Consider the graph from the previous problem, with the order of the vertices being $abcdefg$.

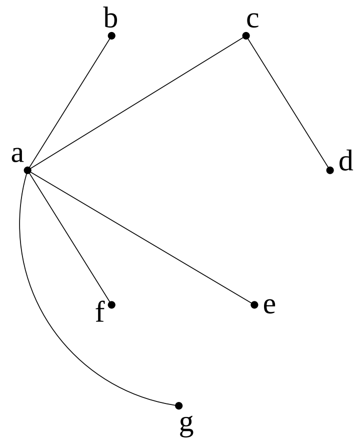



(a) (5 points) Find a spanning tree using the breadth-first algorithm. Show the intermediate steps in finding the tree, by writing down the edge added at each step.

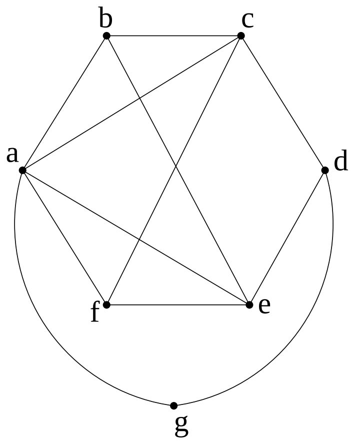
We start at $a$, then add the following edges in order:

$$(a, b), (a, c), (a, e), (a, f), (a, g), (c, d)$$

with the result being the spanning tree



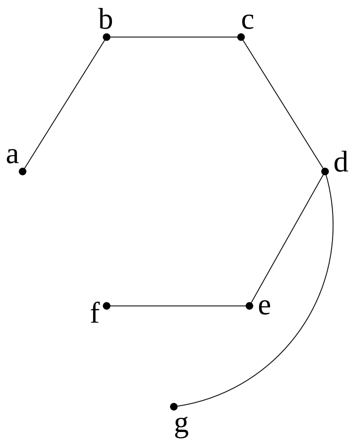
(b) (5 points) Find a spanning tree using the depth-first algorithm. Show the intermediate steps in finding the tree, by writing down the edge added at each step.



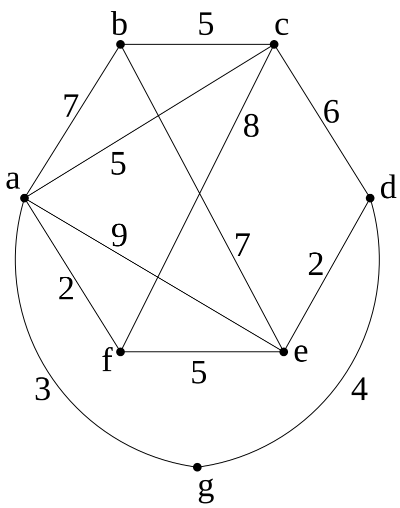

We start at $a$, then add the following edges in order:

$$(a, b), (b, c), (c, d), (d, e), (e, f), (d, g)$$

with the result being the spanning tree



(c) (5 points) We put weights on the edges of the graph as follows:

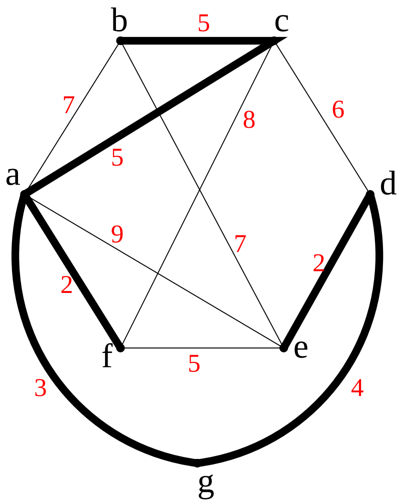



Find a minimal spanning tree using Prim's algorithm. Show the intermediate steps by writing down the edge added at each step.

We start at $a$, then add the following edges in order:

$$(a, f), (a, g), (g, d), (d, e), (a, c), (c, b)$$

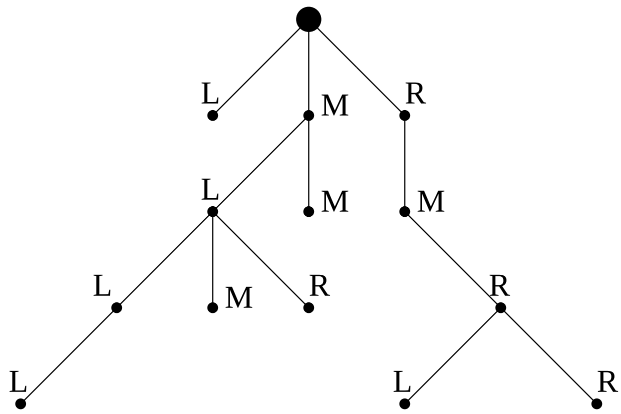
with the result being the spanning tree in boldface:



**7.** We define a *ternary tree* to be a rooted tree with the following properties:

- Each vertex has at most three children;
- Each child of a vertex is labeled by $L$ (left), $M$ (middle) or $R$ (right), and no two children of the same vertex have the same label.

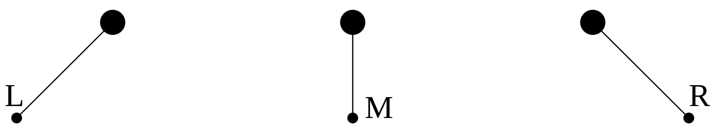
An example is shown below:




We define an isomorphism of ternary trees to be a bijection between their sets of vertices with the following properties:

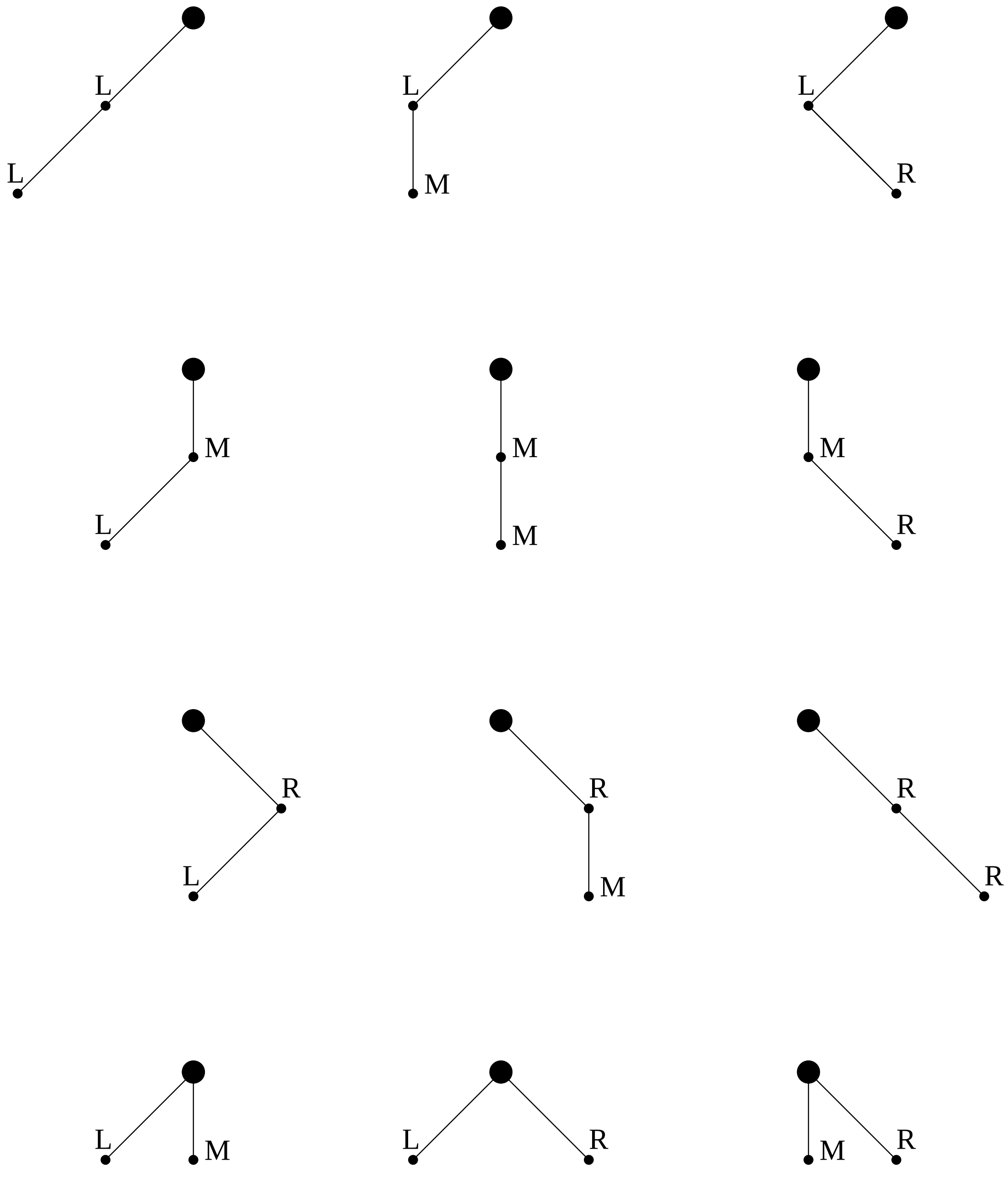
- It preserves the adjacency relation;
- It takes the root to the root;
- It preserves the labels: it takes $L$ children to $L$ children, $M$ children to $M$ children, and $R$ children to $R$ children.

For example, these are all nonisomorphic ternary trees with 2 vertices:




(Continued on next page)

(a) (3 points) Draw all nonisomorphic ternary trees with 3 vertices.



(b) (3 points) Let $a_n$ be the number of nonisomorphic ternary trees with $n$ vertices. Find a recurrence relation for $a_n$.

A ternary graph with $n$ vertices is composed of its root and three subtrees coming from the root: left, middle, and right. If $i, j, k$ denote the number of vertices in the left, middle, and right subtrees, respectively, then $i+j+k = n-1$. Each subtree is a ternary graph itself. Therefore,

$$a_n = \sum_{i+j+k=n-1} a_i a_j a_k.$$

(c) (4 points) Let $h$ be the height of a ternary tree with $n$ vertices. Show that:

$$h \geq \lceil \log_3(2n+1) \rceil - 1.$$

At level 0 we have at most 1 vertex, at level 1 at most 3 vertices, at level 2 at most $3^2$ vertices, . . . , at level $h$ at most $3^h$ vertices. Thus, the total number of vertices $n$ is at most

$$1 + 3 + 3^3 + \cdots + 3^h = \frac{3^{h+1} - 1}{3 - 1}.$$

The inequality

$$n \leq (3^{h+1} - 1)/2$$

can be rewritten as

$$\log_3(2n+1) \leq h + 1$$

or

$$h \geq \log_3(2n+1) - 1.$$

Since $h$ is an integer, we can use the ceiling of $\log_3(2n+1)$.

*Do not write on this page.*

| | | |
|---|---|---|
| 1 | | out of 20 points |
| 2 | | out of 25 points |
| 3 | | out of 10 points |
| 4 | | out of 10 points |
| 5 | | out of 10 points |
| 6 | | out of 15 points |
| 7 | | out of 10 points |
| Total | | out of 100 points |