

Math 61 Midterm 2
November 20, 2015

50 minutes

Your Name: _____

UCLA ID: _____

SECTION: Cross one box below

Day \ T.A.	John	Zach	Sam
Tuesday	(1A)	1C	1E
Thursday	1B	1D	1F

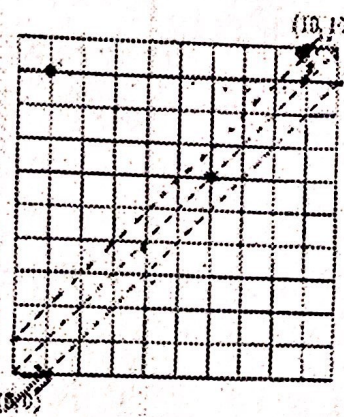
Rules: You MUST simplify completely and BOX all answers with an INK PEN. You are allowed to use only this paper and pen/pencil. A one-sided hand-written formula sheet is allowed. No calculators, no books, no notebooks, no web access. You MUST write your name and UCLA id. Except for the last problem, you MUST write out your logical reasoning and/or proof in full. You have exactly 50 minutes.

Warning: At 1:50pm your out of time, those caught writing after time get automatic 10% score deduction.

Problem	Value	Score
Problem 1	12	7
Problem 2	8	4
Problem 3	8	2
Problem 4	10	9
Problem 5	12	7
Total	50	29

Problem 1. How many grid paths are there from $(0,0)$ to $(10,10)$ that

- (a) Stay on or above the diagonal $y = x$ and go through $(1,9)$.
- (b) Go through $(6,6)$ and from that point on stay on or above the diagonal $y = x$.
- (c) Only touch the diagonal $y = x$ at $(0,0)$ and $(10,10)$.
- (d) Stay on or above the diagonal $y = x - 1$.



Note: If needed you can use the formula for Catalan numbers $C_n = \frac{1}{n+1} \binom{2n}{n}$ or write your answers in terms of C_n .

~~a)~~ ~~C_9~~

$C_1 \cdot C_9 = C_9$

$(0,0) \rightarrow (1,9)$

$C_9 + 4$

b). $\binom{12}{6} \times C_4$ ✓

~~c)~~ $2 \times C_{10}$ ✓ C_{10}

d). we used the catalan number for $(11 \times 11 \text{ grid})$
 C_{11} ✓

Problem 2.

(a) Find a recurrence and initial conditions for a_n for $n \geq 0$ with solution

$$a_n = \frac{1}{2}(1 + \sqrt{2})^n + \frac{1}{2}(1 - \sqrt{2})^n,$$

(b) Show that $\frac{1}{2}(1 + \sqrt{2})^n + \frac{1}{2}(1 - \sqrt{2})^n$ is an integer for all $n \geq 0$.

a). $1/r_1^n \rightarrow 1/r_2^n$

two roots $(1 + \sqrt{2})$ and $(1 - \sqrt{2})$

$$(x - (1 + \sqrt{2}))(x - (1 - \sqrt{2}))$$

$$= x^2 - (1 - \sqrt{2})x + (1 + \sqrt{2})x - 1$$

$$= x^2 + 2\sqrt{2}x - 1$$

$$a_n = -2\sqrt{2}a_{n-1} + a_{n-2}$$

initial condition:

$$a_0 = 1$$

$$a_1 = \frac{1}{2}(1 + \sqrt{2}) + \frac{1}{2}(1 - \sqrt{2})$$

$$= 1$$

b). prove true for $n=0$

$$\frac{1}{2} + \frac{1}{2} = 1$$

Assume true for $n=k$, prove for $n=k+1$

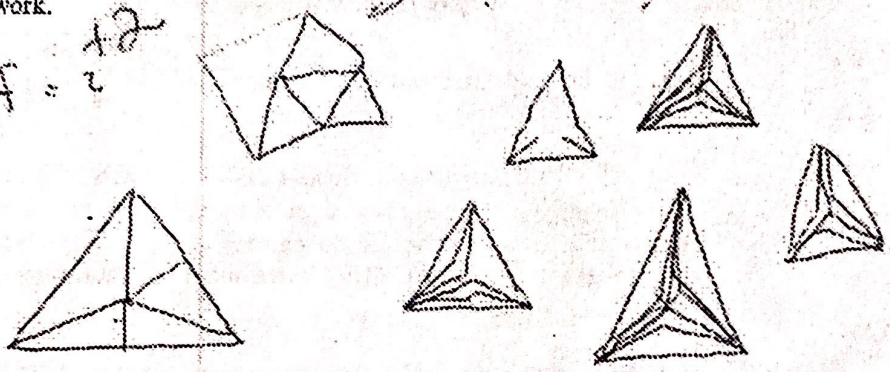
$$\frac{1}{2}(1 + \sqrt{2})^{k+1} + \frac{1}{2}(1 - \sqrt{2})^{k+1}$$

$$= \frac{1}{2}(1 + \sqrt{2})^k \cdot (1 + \sqrt{2}) + \frac{1}{2}(1 - \sqrt{2})^k \cdot (1 - \sqrt{2})$$

Problem 3. In a planar graph G all vertices have degree 5 and all faces have degree 3 (i.e. faces are triangles).

- (a) Relate the number of vertices $|V|$ with the number of edges $|E|$ of G .
 (b) Relate the number of faces f of a planar embedding of G with the number of edges $|E|$ of G .
 (c) Use Euler's formula to find the number of vertices $|V| = \underline{\hspace{2cm}}$, edges $|E| = \underline{\hspace{2cm}}$ and faces $f = \underline{\hspace{2cm}}$ of G .
 You must show your work.

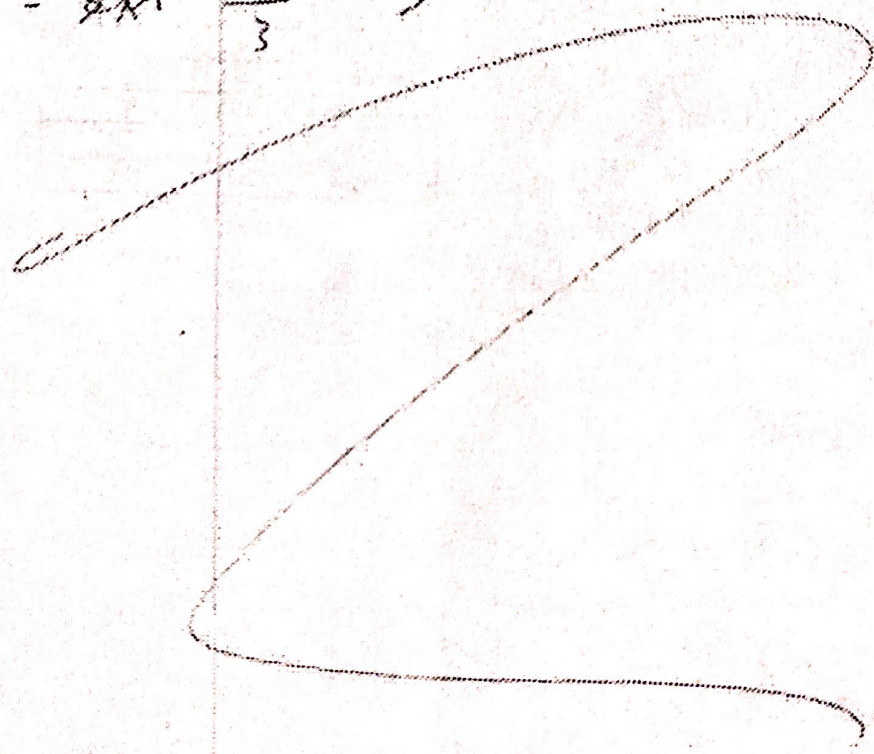
a) $|V| - |E| + f = 2$



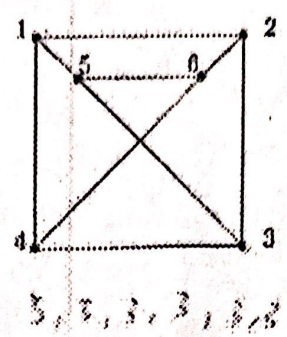
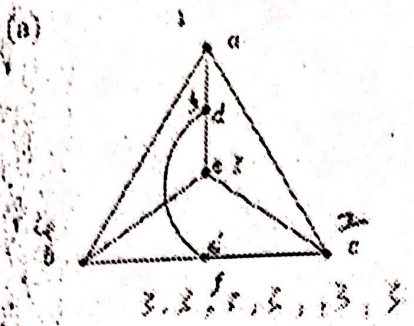
$3|V| = 2|E|$

b) $f = \frac{|E|}{3} + 2$

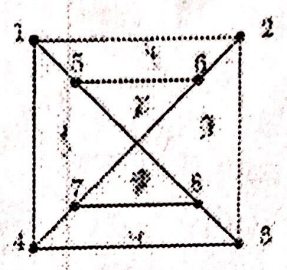
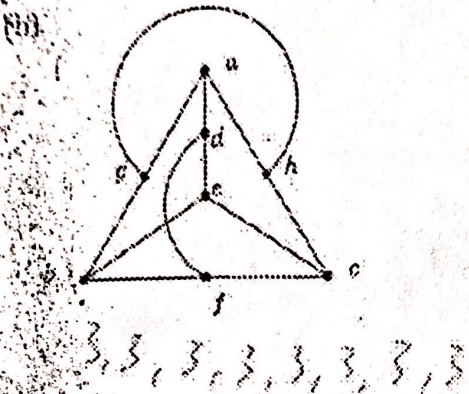
(c) 2/8



Problem 4. Decide whether the following pairs of graphs are isomorphic or non-isomorphic.



- a → 1
- b → 4
- c → 2
- d → 5
- e → 3
- f → 6



- a →
- b →
- c →
- d →
- e →
- f →
- g →
- h →

Note: In case of isomorphism you must write a bijection in the figure above right (using ink). No need of further argument. In case of non-isomorphism you must say so and present an argument why the two graphs are not isomorphic.

(a) Isomorphic

(b) Not isomorphic. neither graph is planar!
 The newly added face has ~~degree~~ degree 5.
 That does not exist in the second diagram. 3-cycles are what you want.

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Problem 5. True or False Circle the answers only with ink, next to the questions. No reasoning/calculations will be taken into account.

- T or F. (1) If a person is paid at the start of every two weeks then there is a month of the year with three payments.
- T or F. (2) $F_n \leq \binom{n-2}{2}$ for all integers $n \geq 1$.
- T or F. (3) The hypercube H_n for all $n \geq 2$ has a closed Eulerian tour.
- T or F. (4) The hypercube H_n for all $n \geq 2$ has a Hamiltonian cycle.
- T or F. (5) A graph isomorphic to a bipartite graph is also bipartite.
- T or F. (6) Graph K_4 is a subgraph of the 4-cube H_4 .
- T or F. (7) Dijkstra's algorithm finds the weight of the shortest Hamiltonian cycle with vertices a and z of a graph.
- T or F. (8) If A is the adjacency matrix of a graph with n vertices then $A_{1,1}^2 + A_{2,2}^2 + \dots + A_{n,n}^2$ is always even.
- T or F. (9) $(2, 2, 2, 2, 4, 4)$ is a valid degree sequence for a simple graph.
- T or F. (10) $(3, 3, 3, 3)$ is a valid degree sequence for a simple graph.
- T or F. (11) All simple graphs with degree sequence $(3, 3, 3, 3, 3, 5)$ are non planar.
- T or F. (12) All simple graphs with degree sequence $(4, 4, 4, 4, 4)$ are non planar.

Handwritten calculations:

$$\frac{14}{3} \sqrt{365}$$

$$\frac{14}{3} \sqrt{365}$$

$$\frac{23}{3} \sqrt{365}$$

$$\frac{23}{3} \sqrt{365}$$
