Midterm

Last Name: [redacted]

First Name: [redacted]

Student ID: [redacted]

Signature: [redacted]

Section:

	Tuesday:	Thursday:	
	1A	1B	TA: KISSLER, CAMERON
	1C	1D	TA: STARK, TALON

Instructions: Please submit your solutions before June 11 8am via gradescope.
Notice that you are permitted to use notes, texts, computers, and standard internet sites (such as Wikipedia) during the exam. **But you are not permitted to use human resources (including Chegg, Math Stack Exchange, etc.) or to collaborate. Violations of these rules will be regarded as academic dishonesty and will be reported to the office of the Dean of Students.**
You must **show your work** to receive credit(if you only show your final results, you will get at most half of the full points). Please circle or box your final answers.
Organize your work, in a reasonably neat and coherent way, otherwise your solutions may not be graded.

Please do not write below this line.

Problem	Max	Score
1	10	
2	10	
3	6	
4	8	
5	5	
6	10	
7	10	
8	15	
9	14	
10	12	
Total	100	

1. Let A be a set of 4 elements and B a set of 3 elements

(a) (2 pts) How many elements in the set $A \times B$?

$|A \times B| = |A| \cdot |B| = 4 \cdot 3 = \boxed{12 \text{ elements}}$

(b) (2 pts) How many 4-element subset does $A \times B$ have?

$\binom{12}{4} = \frac{12!}{4!(8)!} = \boxed{495 \text{ 4-element subsets}}$

How many combinations of 4 can be made out of 12

(c) (4 pts) How many onto functions are there from A to B?

For each element y in B, there must be an element x in A where $f(x) = y$

There are $\binom{4}{2} = 6$ ways to choose two elements in A to map to one element in B. There are three ways to choose which element in B they should map to, and two ways to order the last element. $6 \times 3 \times 2 = \boxed{36 \text{ onto functions}}$

(d) (2 pts) How many one-to-one functions are there from B to A?

For each element y in B, there must be exactly one element in A such that $f(y) = A$

$4 \cdot 3 \cdot 2 = \boxed{24 \text{ one-to-one functions}}$

2. (a) (4 pts) For propositions p, q, and r use the truth table to prove

$$p \to (q \to r) \equiv \neg p \vee (\neg r \to \neg q)$$

p	q	r	$\neg p$	$\neg q$	$\neg r$	$q \to r$	$\neg r \to \neg q$	$p \to (q \to r)$	$\neg p \vee (\neg r \to \neg q)$
T	T	T	F	F	F	T	T	T	T
T	T	F	F	F	T	F	F	T	T
T	F	T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T	T	T
F	T	T	T	F	F	T	T	T	T
F	T	F	T	F	T	F	F	F	F
F	F	T	T	T	F	T	T	T	T
F	F	F	T	T	T	T	T	T	T

(b) (2 pts) State the negation of the the proposition: $(\forall n \in \mathbb{N})\sqrt{n}$ is either an integer or an irrational number.

$(\forall n \in \mathbb{N}) \sqrt{n} \in \mathbb{Z} \mid \sqrt{n} \in \mathbb{R} \setminus \mathbb{Q}$

~~$\neg [(\forall n \in \mathbb{N}) \sqrt{n} \in \mathbb{Z} \mid \sqrt{n} \in \mathbb{R} \setminus \mathbb{Q}]$~~

$\boxed{(\exists n \in \mathbb{N}) \sqrt{n} \notin \mathbb{Z} \ \& \ \sqrt{n} \notin \mathbb{R} \setminus \mathbb{Q}}$

(c) (4 pts) State the negation of the proposition: $(\exists n \in \mathbb{N}) 2^n \leqslant n$. Prove that this negation is true. Hint: use mathematical induction.

For base cases, $n=0, n=1$, $\neg [(\exists n \in \mathbb{N}) 2^n \leq n]$

$(\forall n \in \mathbb{N}) 2^n > n$

$n=0$ $2^0 > 0$

$1 > 0$ ✓

$n=1$ $2^1 > 1$

$2 > 1$ ✓

For the inductive case, assume statement holds for $n=k$,

$(\forall n \in \mathbb{N}) 2^k > n$

$2^{k+1} > n$

$2^k \cdot 2 > n$

$2^k \cdot 2 > 2^k > n$ ✓

-2

Incorrect Induction step

Therefore, by Mathematical Induction, the statement $(\forall n \in \mathbb{N}) 2^n > n$ is true

3. Solve the following recurrence relations.

(a) (3 pts) $a_n = 3a_{n-1} - a_{n-2}, a_1 = 1, a_2 = 3.$

$a_n - 3_{n-1} + a_{n-2} = 0$

guess $a_n = t^n$

$t^n - 3t^{n-1} + t^{n-2} = 0$

$t^2 - 3t + 1 = 0$

$t = \frac{3 \pm \sqrt{5}}{2}$

$a_n = b\left(\frac{3+\sqrt{5}}{2}\right) + d\left(\frac{3-\sqrt{5}}{2}\right)$

$\boxed{a_n = \frac{\sqrt{5}}{5}\left(\frac{3+\sqrt{5}}{2}\right)^n - \frac{\sqrt{5}}{5}\left(\frac{3-\sqrt{5}}{2}\right)^n}$

$a_1 = \left(\frac{3+\sqrt{5}}{2}\right)b + \left(\frac{3-\sqrt{5}}{2}\right)d = 1$

$\frac{3}{2}b + \frac{\sqrt{5}}{2}b + \frac{3}{2}d - \frac{\sqrt{5}}{2}d = 1$

$3b + \sqrt{5}b + 3d - \sqrt{5}d = 2$

$b = \frac{2 + \sqrt{5}d - 3d}{3 + \sqrt{5}}$

$b = \frac{(3-\sqrt{5})(2 + (\sqrt{5}-3)d)}{4}$

$b = \frac{(3-\sqrt{5})(2 + \sqrt{5}(-\frac{\sqrt{5}}{5}) - 3(-\frac{\sqrt{5}}{5}))}{}$

$b = \frac{(3-\sqrt{5})(1 + \frac{3\sqrt{5}}{5})}{}$

$b = \frac{\cancel{3} + \frac{9\sqrt{5}}{5} - \sqrt{5} - \cancel{3}}{4}$

$b = \frac{4\sqrt{5}}{5 \cdot 4}$

$b = \frac{\sqrt{5}}{5}$

$a_2 = b\left(\frac{9+5+6\sqrt{5}}{4}\right) + d\left(\frac{9+5-6\sqrt{5}}{4}\right) = 3$

$(14+6\sqrt{5})(3-\sqrt{5})(2+\sqrt{5}d-3d) + 4(14-6\sqrt{5})d = 48$

$(6+3\sqrt{5}d-9d-2\sqrt{5}-5d+3\sqrt{5}d)(14+6\sqrt{5}) + 56-24\sqrt{5}d = 48$

$(6-14d+6\sqrt{5}d-2\sqrt{5})(14+6\sqrt{5}) + 56d - 24\sqrt{5}d = 48$

$84 + 36\sqrt{5} - 196d - 84\sqrt{5}d + 84\sqrt{5}d + 180d - 28\sqrt{5} - 60 + 56d - 24\sqrt{5}d = 48$

$24 + 8\sqrt{5} + 40d - 24\sqrt{5}d = 48$

$d(40 - 24\sqrt{5}) = 24 - 8\sqrt{5}$

$d = \frac{24 - 8\sqrt{5}}{40 - 24\sqrt{5}}$

$d = \frac{3-\sqrt{5}}{5-3\sqrt{5}} \cdot \frac{5+3\sqrt{5}}{5+3\sqrt{5}}$

$d = \frac{15 + 9\sqrt{5} - 5\sqrt{5} - 15}{25 - 45}$

$d = \frac{4\sqrt{5}}{-20}$

$d = \frac{-\sqrt{5}}{5}$

(b) (3 pts) $a_n = 4a_{n-1} - 4a_{n-2}, a_1 = 3, a_2 = 5.$

$a_n - 4a_{n-1} + 4a_{n-2}$

guess $a_n = t^n$

$t^n - 4t^{n-1} + 4^{n-2} = 0$

$t^2 - 4t + 4 = 0$

$(t-2)^2 = 0$

$t = 2$

$a_n = b(2)^n + dn(2)^n$

$a_1 = 2b + 2d = 3$

$b = \frac{3-2d}{2}$

$b = \frac{3+\frac{1}{2}}{2}$

$b = \frac{7}{4}$

$a_2 = 4b + 8d = 5$

$6 - 4d + 8d = 5$

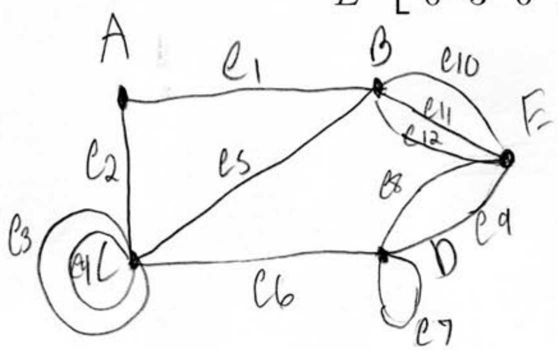
$4d = -1$

$d = -\frac{1}{4}$

$\boxed{a_n = \frac{7}{4}(2)^n - \frac{1}{4}n(2)^n}$

4. (a) (4 pts) Draw the graph represented by the adjacency matrix:

$$
\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}
\begin{array}{c}
\begin{array}{ccccc} A & B & C & D & E \end{array} \\
\begin{bmatrix}
0 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 3 \\
1 & 1 & 4 & 1 & 0 \\
0 & 0 & 1 & 2 & 2 \\
0 & 3 & 0 & 2 & 0
\end{bmatrix}
\end{array}
$$



(b) (4 pts) Please label the edges for the graph obtained in part (a). Find the corresponding incidence matrix for the graph in part (a) with your labeled edges.

$$
\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}
\begin{array}{c}
\begin{array}{cccccccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} & e_{11} & e_{12} \end{array} \\
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}
\end{array}
$$

5. (5 pts) Let $S = \mathbb{R} \setminus \mathbb{Q}$. Consider the following relation $\sim$ on S: $x \sim y$ if and only if $\frac{x}{y} \in \mathbb{Q}$. Is $\sim$ an equivalence relation? Prove your conclusion.

<u>Reflexsive</u>

Let $x \in S$

If $x = y$, $\frac{x}{y} = \frac{x}{x} = 1 \in \mathbb{Q}$ and $x \sim x$

Therefore, $(\forall x \in S)\ x \sim x$, and $\sim$ is reflexsive

$\sim$ is reflexsive, symmetric, and transitive; therefore it is an equivalence relation.

<u>Symmetric</u>

Let $x, y \in S$ and $x \sim y$

then, $\frac{x}{y} \in S$

A reciprocal of a fraction is still a fraction, so

$\frac{y}{x} \in S$ and $y \sim x$

Therefore, if $x \sim y$, then $y \sim x$, and $\sim$ is symmetric

<u>Transitive</u>

Let $x, y, z \in S$ and $x \sim y$ and $y \sim z$

If $x \sim y$, then $\frac{x}{y} \in \mathbb{Q}$

this implies that there exists some $i \in \mathbb{Q}$ such that $x = yi$

If $y \sim z$, then $\frac{y}{z} \in \mathbb{Q}$

this implies that there exists some $j \in \mathbb{Q}$ such that $y = zj$

The product of two rational numbers is rational

then, $\frac{x}{i} = y = zj \rightarrow x = zij$

Since $i, j \in \mathbb{Q}$, $ij \in \mathbb{Q}$, and $\frac{x}{z} = ij \in \mathbb{Q}$

Therefore, if $x \sim y$ and $y \sim z$, then $x \sim z$, and $\sim$ is transitive

6. (10 pts) Consider the weighted graph in Figure 1. Find the lengths of a shortest path from vertex E to other vertexes by Dijkstra's algorithm.

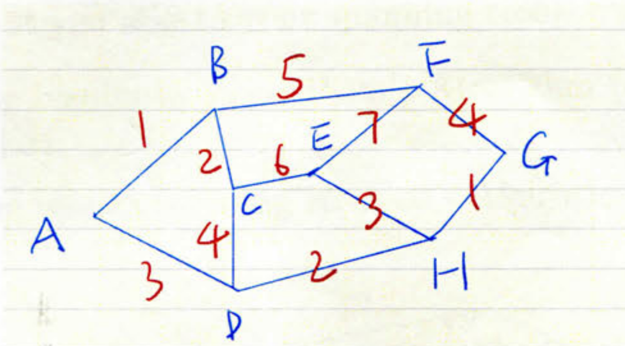


Figure 1: Graph for Question 6.

Dijkstra's Algorithm

Vertex	Weight	Path
A	∞	N/A
B	∞	N/A
C	6	EC
D	∞	N/A
E	0	E
F	7	EF
G	∞	N/A
H	3	EH

V	W	P
A	∞	N/A
B	∞	N/A
C	6	EC
D	5	EHD
E	0	E
F	7	EF
G	4	EHG
H	3	EH

V	W	P
A	∞	N/A
B	∞	N/A
C	6	EC
D	5	EHD
E	0	E
F	7	EF
G	4	EHG
H	3	EH

V	W	P
A	8	EHDA
B	∞	N/A
C	6	EC
D	5	EHD
E	0	E
F	7	EF
G	4	EHG
H	3	EH

V	W	P
A	8	EHDA
B	8	ECB
C	6	EC
D	5	EHD
E	0	E
F	7	EF
G	4	EHG
H	3	EH

V	W	P
A	8	EHDA
B	8	ECB
C	6	EC
D	5	EHD
E	0	E
F	7	EF
G	4	EHG
H	3	EH

V	W	P
A	8	EHDA
B	8	ECB
C	6	EC
D	5	EHD
E	0	E
F	7	EF
G	4	EHG
H	3	EH

V	W	P
A	8	EHDA
B	8	ECB
C	6	EC
D	5	EHD
E	0	E
F	7	EF
G	4	EHG
H	3	EH

7. Use Prim's Algorithm and Kruskal's Algorithm to find the minimal spanning tree for the weighted graph in Figure 2.(Please write down the order of the edges that you add to your spanning tree).

   (a) (5 pts) The results by using Prim's Algorithm (consider the alphabetical order).

   (b) (5 pts) The results by using Kruskal's Algorithm

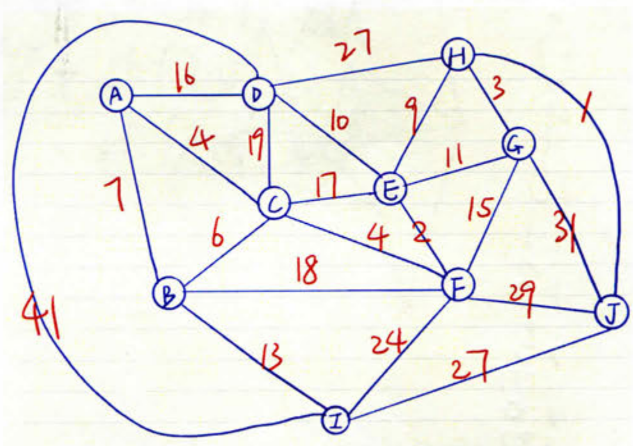
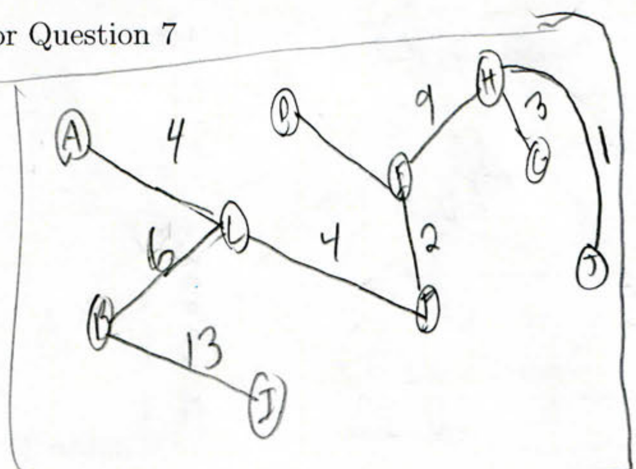


Figure 2: Graph for Question 7

a. Prim's Algorithm

Add the root A to V'
Add C to V', (A,C) to E'
Add F to V', (C,F) to E'
Add E to V', (E,F) to E'
Add B to V', (B,C) to E'
Add H to V', (E,H) to E'
Add J to V', (H,J) to E'
Add G to V', (G,H) to E'
Add D to V', (D,E) to E'
Add I to V', (B,I) to E'



V' = {A, C, F, E, B, H, J, G, D, I}

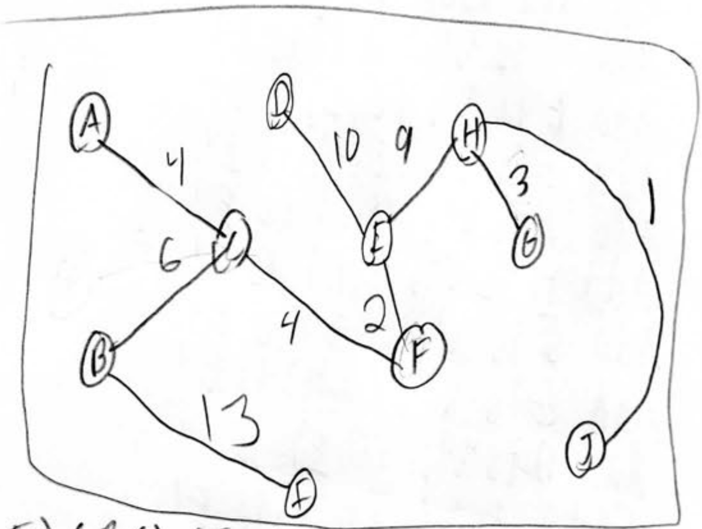
E' = {(A,C), (C,F), (E,F), (B,C), (E,H), (H,J), (G,H), (D,E), (B,I)}

b. Kruskal's Algorithm

Edges in Increasing Order of Weight

Edge	Weight
-(H,J)	1
-(E,F)	2
-(G,H)	3
-(A,C)	4
-(C,F)	4
-(B,C)	6
(A,B)	7
-(E,H)	9
-(D,E)	10
(E,G)	11
-(B,I)	13
(F,G)	15
(A,D)	16
(C,E)	17
(B,F)	18
(C,D)	19
(E,I)	24
(D,H)	27
(I,J)	27
(F,J)	29
(G,J)	31
(D,I)	41

Add (H,J) to E', H and J to V'
Add (E,F) to E', E and F to V'
Add (G,H) to E', G to V'
Add (A,C) to E', A and C to V'
Add (C,F) to E'
Add (B,C) to E', B to V'
Add (E,H) to E'
Add (D,E) to E', D to V'
Add (B,I) to E', I to V'



E' = {(H,J), (E,F), (G,H), (A,C), (C,F), (B,C), (E,H), (D,E) (B,I)}

V' = {H, J, E, F, G, A, C, B, D, I}

8. Determine whether each graph is planar. If the graph is planar, redraw it so that no edges cross; otherwise, find a subgraph homeomorphic to either K_5 or $K_{3,3}$.

(a) (10 pts) See Figure in 3.

(b) (5 pts) See Figure in 4

a. Assume the graph is planar
Each face is bounded by at least three edges
Each edge belongs to at most two faces

$2e \geq 3f$

$2e \geq 3(e-v+2)$

$2e \geq 3e-3v+6$

$2(20) \geq 3(20)-3(12)+6$

$40 \geq 30$ ✓

So the graph may or may not be planar

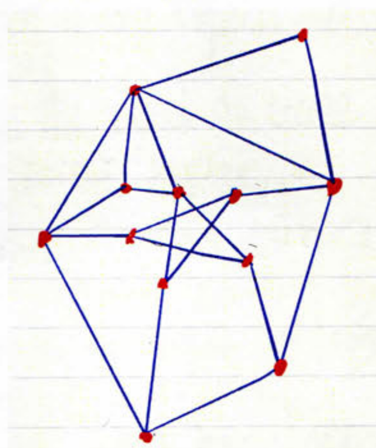


Figure 3: Graph for Question 8(a)

b. Assume the graph is planar
Each face is bounded by at least three edges
Each edge belongs to at most two faces

$2e \geq 3f$

$2e \geq 3(e-v+2)$

$2e \geq 3e-3v+6$

$2(11) \geq 3(11)-3(6)+6$

$22 \geq 21$ ✓

So the graph may or may not be planar

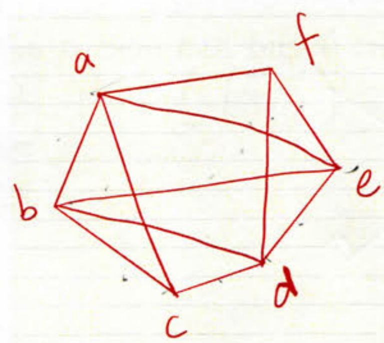
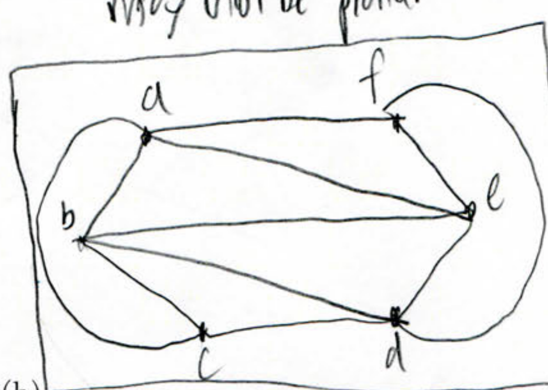
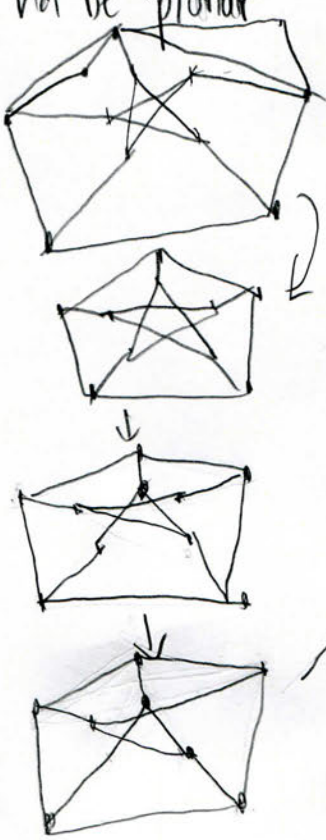
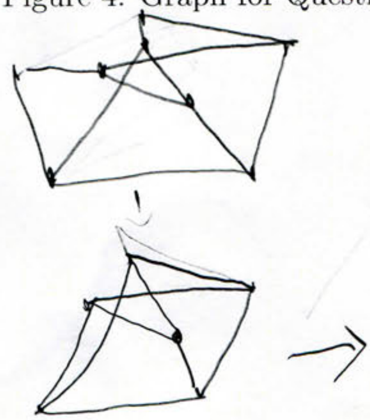
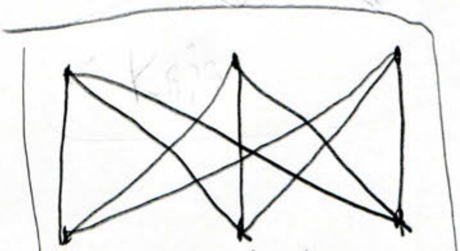


Figure 4: Graph for Question 8(b)









homeomorphic to $K_{3,3}$

9. (a) (10 pts) How many 11-digit binary sequences (i.e., each digit is either a 0 or a 1) do not contain two consecutive 0's?

The number of n-digit binary sequences that do not contain two consecutive 0's can be represented as a recurrence relation

Let a_{n-1} represent the number of binary sequences of this type that end with a 1 and are n-1 bits long

Let b_{n-1} represent the number of binary sequences of this type that end with a 0. and are n-1 bits long

Because a 0 or 1 can be added to create a sequence of length n, and two consecutive 0's cannot appear,

$a_n = a_{n-1} + b_{n-1} = a_{n-1} + a_{n-2}$

$b_n = a_{n-1}$ $\quad b_{n-1} = a_{n-2}$

Thus, the total number of sequences of length n of this type are

$a_n + b_n = 2a_{n-1} + a_{n-2} = a_n + a_{n-1}$

Since $a_1 = 1$ and $a_2 = 2$ and $b_1 = 1$ and $b_2 = 1$, $a_1+b_1 = 2$ and $a_2+b_2 = 3$

This is the Fibonacci Sequence offset by 2. So the number of 11-digit binary sequences with no consecutive zeros is

$\boxed{F_{11+2} = F_{13} = 233 \text{ sequences}}$

(b) (4 pts) A bakery produces 4 kinds of cookies (suppose there are infinitely many of each). A person wants to buy 6 cookies. Find the number of ways the person can buy 6 cookies.

$C(6+4-1, 4-1) = C(9,3) = \boxed{84 \text{ ways}}$

unordered combinations

10. (a) (6 pts) Choose 150 integers from this list $\{1, 2, \cdots, 298\}$, prove that there are two integers n_1, n_2 such that $n_1 | n_2$ or $n_2 | n_1$.

Each integer $n_i \in \{1, 2, \dots, 198\}$, $1 \le i \le 100$, can be represented in the form $n_i = a \cdot 2^b$, where a is an odd integer less than 298, and b is an integer.

If two numbers, $n_1, n_2 \in \{1, 2, \dots, 298\}$ have the same value of a, either $n_1 | n_2$ or $n_2 | n_1$, or both

We can model this mapping as a function $f(n) = a$ from $X = \{150$ integers from $\{1, 2, \dots, 298\}\}$ to $Y = \{1, 3, 5, \dots, 297\}$

Because there are 150 integers chosen, $|X| = 150$

Because there are 149 odd integers in the interval $[1, 298]$, $|Y| = 149$

By the Pigeonhole Principle, there must be at least $\left\lceil \frac{|X|}{|Y|} \right\rceil = \left\lceil \frac{150}{149} \right\rceil = 2$ distinct values n_1, n_2 such that $f(n_1) = f(n_2)$. Therefore, there must be two integers n_1, n_2, with the same value of a, meaning either $n_1 | n_2$ or $n_2 | n_1$

(b) (6 pts) Let $n_1, n_2, \cdots, n_{201}$ be integers. Prove there exist three integers $n_i, n_j, n_k \in \{n_1, n_2, \cdots, n_{201}\}$ such that 100 can divide the differences between any two of them.

Each integer $n_1, n_2, \dots, n_{201}$ can be represented as $100a + r$, where $a, r \in \mathbb{Z}$

If 100 divides two of the integers' difference, $100 | (n_x - n_y)$, where $n_x, n_y \in \{n_1, n_2, n_{201}\}$, the two numbers have the same value for r.

We can model this relationship with a function $f(n) = r$ from $X = \{n_1, n_2, \dots, n_{201}\}$ to $Y = \{0, 1, 2, \dots, 99\}$

Because there are 201 numbers, $|X| = 201$

Because there are 100 nonnegative integers less than 100, so $|Y| = 100$

By the Pigeonhole Principle, there must be at least $\left\lceil \frac{|X|}{|Y|} \right\rceil = \left\lceil \frac{201}{100} \right\rceil = 3$ distinct values n_i, n_j, n_k, such that $f(n_i) = f(n_j) = f(n_k)$. Therefore, there are three integers with the same value of r, where 100 can divide the differences between any two of them.