Math 61
Summer 2017, Session A
Midterm Exam
July 12, 2017
Time Limit: 110 Minutes

Name (Last, First): [redacted]

UCLA ID Number: [redacted]

This exam contains 9 pages (including this cover page) and 8 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

Each problem is worth 25 points. Some problems are broken down into subproblems. The weight of each subproblem is indicated next to the subproblem.

You may **not** use your books, notes, or any calculator on this exam.

The following rules apply:

- **Make shown work clear and easy to find.** It is important for assessing credit that any part of your solution be clearly visible.
- **Unless otherwise stated, you must show your work to obtain full credit on a problem.** A correct answer without appropriate explanation / work will receive a lower score. On the other hand, incorrect answers that show progress may receive partial credit.
- If you need more space, use the back of the pages. If you put work on the back of a page that is part of your final answer for a problem, indicate this clearly. You are encouraged to use the back of the pages as scratchwork. If you need extra scatchpaper, I will have some at the front. Turn in any extra scratchwork with the exam.

Do not write in the table to the right.

Problem	Points	Score
1	25	25
2	25	25
3	25	25
4	25	25
5	25	25
6	25	25
7	25	22
8	25	15
Total:	200	187

1. (25 points) Prove $\sum_{k=1}^{n} k(k!) = (n+1)! - 1$ for $n \geq 1$ by induction (recall $\sum_{k=1}^{n} k(k!) = 1(1!) + 2(2!) + \cdots + n(n!)$).

25

Base Case: $n=1$

$\sum_{k=1}^{(1)} k(k!)$ <LHS> $((1)+1)! - 1$ <RHS>

$= 1 \cdot 1!$ $= 2! - 1$

$= 1$ $= 1$

LHS = RHS

$\therefore \boxed{P(1) \text{ is true}}$

Inductive Step: Assume $P(n)$ is true

$$P(n+1) = \sum_{k=1}^{n+1} k(k!)$$
$$= \sum_{k=1}^{n} k(k!) + (n+1)(n+1)!$$
$$= (n+1)! - 1 + (n+1)(n+1)!$$
$$= (n+1+1)(n+1)! - 1$$
$$= (n+2)(n+1)! - 1$$
$$= (n+2)! - 1$$

$\therefore \boxed{P(n) \rightarrow P(n+1)}$

Because $P(1)$ is true & $P(n) \rightarrow P(n+1)$, by induction,

$$\sum_{k=1}^{n} k(k!) = (n+1)! - 1, \ n \geq 1$$

is true

2. (25 points) Consider the set $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. You need not show your work on this problem.

(a) (5 points) Write any partition of X consisting of at least 3 sets.

$\boxed{S = \{\{1\}, \{2, 3\}, \{4, 5, 6, 7, 8, 9\}\}}$

5

(b) (5 points) How many subsets of X have exactly 6 elements?

$\binom{9}{6} = \boxed{\frac{9!}{3!6!}}$

5

(c) (5 points) How many ordered strings of all elements of X, without repeats, can be formed?

$\boxed{9!}$

5

(d) (5 points) How many ordered strings of 4 elements of X, without repeats, can be formed?

$\boxed{\binom{9}{4} \cdot 4!}$

5

(e) (5 points) How many ordered strings of 6 elements of X, without repeats, can be formed if the ordered strings must begin with either a 2 or an 8?

$\boxed{2 \cdot \binom{8}{5} \cdot 5!}$

begin w/ 2 or 8

# of combinations of remaining part of X.

5

3. (25 points) In this problem, X and Y are finite, non-empty sets and $f: X \to Y$ is a function. Recall $|X|$ denotes the size of X and $|Y|$ denotes the size of Y.

(a) (5 points) Suppose $|X| > |Y|$. State the pigeonhole principle in terms of f in this case.

there is some $x \in X$ that maps to the same $y \in Y$.

b/c... if $x \in X$ maps to unique $y \in Y$

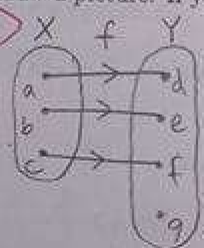
we have $|X|$ unique $x \in X$

then we must have $|X|$ unique $y \in Y$

BUT $|X|$ pigeons cannot fit in $|Y|$ pigeons $(|X| > |Y|)$,

$\therefore$ $\boxed{\text{two } x \in X \text{ must map to one } y \in Y}$

(b) (5 points) Give an example of X, Y, and f where $|X| < |Y|$ and f is one-to-one. You may draw a picture. If you do, make sure your drawing is clear.



$X = \{a, b, c\}$, $Y = \{d, e, f, g\}$, f described in picture.

$\boxed{f \text{ is one-to-one}}$

(for all $x \in X$ f maps to a unique $y \in Y$)

But $|Y| = 4$ and $|X| = 3$

$\therefore$ $\boxed{|X| < |Y|}$

(c) (15 points) Show if $|X| = |Y|$ and f is one-to-one, then f is onto.

set of all elements mapped to by f ↓

suppose $|X| = |Y|$ & f is one-to-one, let $D = \{f(x) \mid x \in X\}$

if f is 1-1, then there exists an unique $x \in X$ for all $d \in D$

$\therefore$ $|X| \geq |D|$

if f is a function, then for all $x \in X$ there exists at least one $d \in D$

$\therefore$ $|X| \leq |D|$

From the above we can deduce

$|X| = |D|$

$\therefore$ $|X| = |Y| = |D|$

$\therefore$ $|Y| = |D|$, by function definition, $D \subseteq Y$

Because $D \subseteq Y$ & $|Y| = |D|$ then for all $d \in D$, $d \in Y$ and for all $y \in Y$, $d \in D$. $\therefore$ $Y = D = \{f(x) \mid x \in X\}$ $\therefore$ $\boxed{f \text{ is onto}}$

4. (25 points) For this problem you may use any result either proven in the book or done in lecture. For part (c) make sure you show work that is more than just citing a theorem.

(a) (5 points) Find the coefficient of a^3x^4 in the expansion of $(a+x)^7$.

25

By Binomial thrm, $(a+x)^n = \sum C(n,k) a^k k^{n-k}$,

for this case, $n=7$, $k=3$

$\therefore$ coeff. is $\boxed{\binom{7}{3}}$ ✓

(b) (5 points) Find the coefficient of $a^3b^2c^3$ in the expansion of $(a+b+c)^8$.

By extending Binomial thrm, we achieve

$(a_0 + a_1 + a_2)^n = \frac{n!}{i!j!(n-i-j)!} a_0^i a_1^j a_2^{n-i-j}$ | $a_0^i a_1^j a_2^{n-i-j}$ span all possible values from algebraic expansion

for this case, $i=3$, $j=2$, $n=8$

$\therefore$ coeff is $\boxed{\frac{8!}{3!2!3!}}$ ✓

(c) (15 points) Find the coefficient of $a^{10}x^7$ in $(a^2+ax+x^2)(a+x)^{15}$.

possible ways to form $a^{10}x^7$

case	(a^2+ax+x^2)	$(a+x)^{15}$
1	a^2	a^8x^7
2	ax	a^9x^6
3	x^2	$a^{10}x^5$

$\therefore$ coeff $= \sum_{\text{cases}}$ coeff of $(a^2+ax+x^2) \cdot$ coeff of $(a+x)^{15}$

$= \boxed{\frac{15!}{8!7!} + \frac{15!}{9!6!} + \frac{15!}{5!10!}}$ by binomal th(part a) ~~extension thrm~~.

✓

5. (25 points) Suppose one has 24 identical slices of cake and 17 identical slices of pie to distribute to 9 party guests.

(a) (5 points) In how many ways can the cake slices be distributed?

9 guests 24 slices of cake
$\downarrow$
9 bins 24 balls (or 8 dividers)

$\therefore \binom{24+9-1}{24}$

$= \boxed{\binom{32}{24}}$

5

(b) (10 points) In how many ways can the cake slices and the pie slices be distributed?

combinations

10

= ways of distributing cake × ways of ... pie

from part A

$= \binom{32}{8} \cdot \binom{17+8}{8}$ 9 bins 17 balls.

(indep events)

$= \boxed{\binom{32}{8} \cdot \binom{25}{8}}$

$\Downarrow$

multiplication rule

(c) (10 points) In how many ways can this be done if each guest must get at least 1 slice of each cake and pie and if no guest gets more than 2 slices of pie?

event ① put 1 cake "ball" in each cake "bin"

$ball_{cake} = 24 - 9 \cdot 1 = 15$

4

event ② subtract # of ways to distribute more than 2 pie "balls", from # of ways to distribute pie "balls"

2a. # of ways to distribute pies where 1 or more guests get >2 pie slices.

~~xx 1 2 1 2 1 2 1 2 1 2 1 2 1 2~~

~~22222~~ on average each guest gets $\frac{17}{9}$ pies.

flip ↗

~~[illegible]~~ . . .

$\therefore$ By pigeonhole if one guest gets 0 pies, one must get >2 pies.

$(2 \cdot 8 = 16 < 17)$.

$\therefore$ ways of distributing pies is... ways of distributing 1-2 pies to each person

$= \binom{9}{17-9} = \binom{9}{8}$

$\therefore$ total # of ways

$=$ event ① $\cdot$ event ②

$= \binom{23}{8} \cdot \binom{9}{8}$

$= \boxed{9\binom{23}{8}}$

event ①.

combination #

$= \binom{15+8}{8}$

$= \binom{23}{8}$.

6. (25 points) Let $X = \{1, 2, 3\}$.

(a) (5 points) List all of the subsets of X.

all subsets:

$\emptyset$
$\{1\}$
$\{1,2\}$
$\{1,3\}$
$\{2,3\}$
$\{2\}$
$\{3\}$
$\{1,2,3\}$

5

Define a relation $\lesssim$ on subsets of X by $A \lesssim B$ if $A \subseteq B$.

(b) (10 points) Circle the following relation properties that are TRUE for $\lesssim$:

Reflexive Symmetric Antisymmetric Transitive.

For any property you did not circle, provide an example of subsets of X demonstrating that the property does not hold.

10

Symmetric

let $B = \{1,2,3\}$; $A = \{1,2\}$

$A \subseteq B$ ✓ A is a subset of B b/c if $x \in A$, then $x \in B$.

$B \subseteq A$ ✗ let $x = 3$, $3 \in B$ but $3 \notin A$

$\therefore A \lesssim B \not\Rightarrow B \lesssim A$

$\therefore$ $\boxed{\text{not symmetric}}$

10

(c) (10 points) Circle the following types of relations that $\lesssim$ is:

Partial Order Total Order Equivalence Relation

$A = \{1,2\}$ $A \not\subseteq B$ Not symm.
$B = \{a,b\}$ $B \not\subseteq A$

7. (25 points) In the following question note that we do NOT work with the same recurrence relation the whole time.

(a) (5 points) Consider the recurrence relation $a_n = 6a_{n-1} - 2a_{n-2} + 7a_{n-3}$ for $n \geq 3$. Find the characteristic equation of this relation. You do not need to show your work on this part.

2

By recurrence relations and linear algebra,

$a_n = 6a_{n-1} - 2a_{n-2} + 7a_{n-3}$, $n \geq 3$

$\downarrow$

$t^n = 6t^{n-1} - 2t^{n-2} + 7t^{n-3}$

$\downarrow$

$t^n - 6t^{n-1} + 2t^{n-2} - 7t^{n-3} = 0$

$\downarrow n=3$

char eq:

$$\boxed{t^3 - 6t^2 + 2t - 7 = 0}$$

✓

(b) (10 points) Suppose a linear recurrence relation has characteristic equation $t^2 - 6t + 9 = 0$. What is the general form of the solution of this equation?

7

$t^2 - 6t + 9 = 0$

$(t-3)(t-3) = 0$

$r = 3.$

By linear alg,

$a_n = Ar_1^n + Bnr_2^n$

if $r_1 = r_2$

general form solution

$$\boxed{A(3)^n + B_n(3)^n}$$

(c) (10 points) Suppose c_n is a linear recurrence relation with $c_0 = 0$, $c_1 = 10$, and general solution $A2^n + B3^n$ where A and B are constants. Find an explicit expression for c_n.

$A2^n + B3^n$ = general solution = c_n

$\underline{c_0}$ $\begin{cases} 0 = A + B \\ 10 = 2A + 3B \end{cases} \rightarrow \begin{matrix} A = -10 \\ B = 10 \end{matrix}$
$\underline{c_1}$

explicit expression:

$$\boxed{c_n = (-10)(2)^n + (10)(3)^n}$$

✓

8. (25 points) Let $X = \{1, 2, \ldots, n\}$. Consider a one-to-one, onto function $f\colon X \to X$. For $i \in \mathbb{Z}^+$, we define $f^i(x) = (f \circ f \circ \cdots \circ f)(x)$ where f is composed with itself i times.

15

(a) (5 points) How many one-to-one functions are there from X to itself? You need not show work on this part.

5

How many ways to map $X \to X$, fix X then map (non repeating (1-1))

if f is fixed: $\boxed{n!}$ ~~wrong~~ | ~~If f is arbitrary function there is~~ $\boxed{n^n}$

(b) (10 points) Show there are integers $j, k \in \mathbb{Z}^+$ with $j < k$ such that $f^j(x) = f^k(x)$ for all $x \in X$.

5

f is 1-1 and onto $\therefore$ each $x \in X$ maps to unique $x \in X$

If above statement is false, there must be no mapping loops.

This is false by induction.

P(n) means $|X| = n$ has $j, k \in \mathbb{Z}^+$ s.t. $f^j(x) = f^k(x)$

Base case $|X| = 2$

$x_1 \xrightarrow{f} x_2$
$x_2 \xrightarrow{f} x_1$ $\therefore$ $x_1 \xrightarrow{f} x_2 \xrightarrow{f} x_1$ where $X = \{x_1, x_2\}$

P(2) is true

By inspection / sim. logic P(1) is too.

Inductive step

Because f is onto, adding an element forces it to connect two loops which forms a new loop.

might have different loops;

Pictorally:

$x_1 \to x_3 \to \cdots$

add in x_3

$x_1 \to x_2 \to \cdots$
$\to x_3 \to \cdots$

$\therefore$ adding an element does not break loop

$\therefore \boxed{P(n) \to P(n+1)}$

$\therefore$ statement in b) is true

(c) (10 points) Show there is an integer $m \in \mathbb{Z}^+$ with $f^m(x) = x$ for all $x \in X$.

5

As seen in b), ~~because~~ there are ~~loop~~ mapping loops.

If c) is true there is a mapping where the n^{th} map all coincide w/ $f^m(x) = x$. i.e. ~~the~~ for $m \in \mathbb{Z}^+$, $f^m(x)$ goes through all possible combinations of x $\therefore$ true.

or more formally...

$m \in \mathbb{Z}^+$, $f^m(x) = x$ exists.

Base case: $|X| = 1$

$X = \{x_0\}$

$f(x_0) = x_0$

$\therefore m = 1 \rightarrow \boxed{P(1) \text{ is true}}$

Inductive Step

If $P(n)$ is true, add another element to X and map it to something and something to it. similar to part a)...

a new loop is formed.

$$x_1 \xrightarrow{f} x_3 \rightarrow \dots \rightarrow x_{n+1} \rightarrow \dots \rightarrow x_1$$

might be disjoint loops here too

(This proves part a)

But the state where the base case is true is still present.

$\therefore \boxed{P(n) \rightarrow P(n+1)}$

$\therefore$ there must exist $f^m(x) = x$ for $m \in \mathbb{Z}^+$