Math 61, Lec 1
Winter 2016
Exam 2
2/22/16
Time Limit: 50 Minutes

Name (Print): ____________________

Name (Sign): ____________________

Discussion Section: ____________________

This exam contains 6 pages (including this cover page) and 5 problems. Check to see if any pages are missing.

You may *not* use books, notes, or any calculator on this exam.

Unless otherwise stated in the problem, you may leave all answers in terms of $\binom{n}{k}$, $P(n,k)$, $k!$, or any sum, difference, product, or quotient of such symbols.

Partial credit will only be awarded to answers for which an explanation and/or work is shown.

Please attempt to organize your work in a reasonably neat and coherent way, in the space provided. If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	24	
2	16	
3	22	
4	18	
5	20	
Total:	100	

1. (24 points) There are 999,999 natural numbers less than one million. We write any of them as a six digit number, including leading zeros. (For example, 001124 is how we write the number 1124).

   (a) How many of these numbers have all different digits?

   Answer: $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5$

   (b) How many of these numbers have exactly four distinct digits? (For example, 922433 is valid, but 922435 is not valid and 922444 is not valid).

   Answer: $C(10,4)\left(C(4,1)\frac{6!}{3!} + C(4,2)\frac{6!}{2!2!}\right)$

   (c) How many of these numbers have digits that sum to 18?

   Answer: $C(23,5) - 6C(13,5)$

2. (16 points) Solve the recurrence relation $A_n = 3A_{n-1} + 4A_{n-2}$, where $A_0 = 3$, $A_1 = 7$.

Answer: Different versions had different initial conditions. For all versions, general form is

$$A_n = b4^n + d(-1)^n.$$

For these initial conditions, we have $A_n = 2(4^n) + (-1)^n$.

3. (22 points) Recall that a k-cycle is a cycle that includes k edges. In this problem, you will prove Ramsey's theorem, which states that if $n \geq 6$ and we color each edge of K_n either blue or red, then there must exist either a set of three blue edges that form a 3-cycle, or a set of three red edges that form a 3-cycle.
To this end, let $n \geq 6$ be arbitrary, and suppose every edge in K_n is colored either blue or red. Let v_1 be a vertex in K_n.

(a) Prove that at least three of the edges incident to v_1 are the same color.

Answer: In K_n, every vertex has degree $n-1$. Thus, since $n \geq 6$, there are at least 5 edges incident to v_1. To color the edges, we map each edge to either red or blue. Then by the 3rd form of Pigeonhole Principle, there are at least $\lceil \frac{n-1}{2} \rceil \geq \lceil \frac{5}{2} \rceil = 3$ edges incident to v_1 of the same color.

(b) In the previous part, you proved that at least three of the edges incident to v_1 are the same color. Without loss of generality, you may assume that color is blue. Suppose that $\{v_1, v_2\}$, $\{v_1, v_3\}$, and $\{v_1, v_4\}$ are blue edges. Prove that between these four vertices, there must exist either a blue 3-cycle or a red 3-cycle.

Answer: Consider the edges $\{v_2, v_3\}$, $\{v_3, v_4\}$, and $\{v_2, v_4\}$. If any of these three edges is blue, that edge will form a blue 3-cycle with v_1. The only other case is if all three of these edges are red, in which case these three edges will form a red 3-cycle.

4. (18 points) Prove the combinatorial identity

$$\sum_{i=0}^{k} C(m+k-i-1, k-i) \cdot C(n+i-1, i) = C(m+n+k-1, k)$$

using a combinatorial argument. No more than half credit will be awarded to an algebraic proof. (Hint: Use Pirates and Gold.)

Answer: Suppose m distinct blue pirates and n distinct green pirates come upon a treasure of k identical pieces of gold. We want to count the number of ways that the gold can be distributed among the pirates. We count this number in two ways.

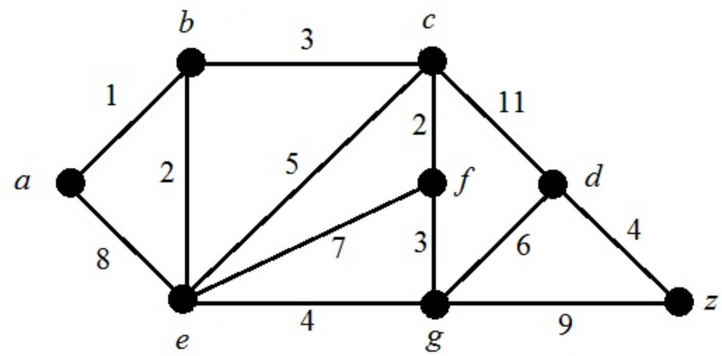
RHS: We consider all $m+n$ pirates together, and count the number of ways to distribute k identical objects to $m+n$ distinct objects. This number is $C(m+n+k-1, k)$.

LHS: We partition the set of outcomes according to how much of the gold goes to the green pirates. For any integer i such that $0 \le i \le k$, the green pirates could get i pieces of the gold, in which case the blue pirates would get the remaining $k-i$ pieces of the gold. If the green pirates get i pieces of the gold, then we count the number of ways to distribute i identical objects to n distinct objects. This number is $C(n+i-1, i)$. If the blue pirates get the remaining $k-i$ pieces of the gold, then we count the number of ways to distribute $k-i$ identical objects to m distinct objects. This number is $C(m+k-i-1, k-i)$.
Thus by multiplication principle, there are $C(m+k-i-1, k-i) \cdot C(n+i-1, i)$ ways to distribute the gold among the pirates such that the green pirates get exactly i pieces. Since this holds for any $0 \le i \le k$, we have that the total number of ways to distribute the gold among the pirates is $\sum_{i=0}^{k} C(m+k-i-1, k-i) \cdot C(n+i-1, i)$.

Since both methods count the same set, we have that

$$\sum_{i=0}^{k} C(m+k-i-1, k-i) \cdot C(n+i-1, i) = C(m+n+k-1, k).$$

5. (20 points) Run Dijkstra's algorithm on the following graph to find the shortest path from a to z. Recall that at each stage of Dijkstra's algorithm, one vertex is chosen and given a permanent label which represents the length of the shortest path from a to that vertex. Write down the list of vertices in the order in which they are given permanent labels. Additionally, find the length of a shortest path from a to z.



Answer: a, b, e, c, f, g, d, z. Shortest path from a to z has length 16.