3. (22 points) Recall that a k-cycle is a cycle that includes k edges. In this problem, you will prove Ramsey's theorem, which states that if $n \geq 6$ and we color each edge of K_n either blue or red, then there must exist either a set of three blue edges that form a 3-cycle, or a set of three red edges that form a 3-cycle.

To this end, let $n \geq 6$ be arbitrary, and suppose every edge in K_n is colored either blue or red. Let v_1 be a vertex in K_n.

2

Prove that at least three of the edges incident to ~~v_1 are the same~~ color. ← this is not what you do.

as $n \geq 6$ then each vertex must have a degree which is $\deg(v_i) \geq 5$ where v_i is arbitrary. Assuming in the worst case where no. of blue edges = no. of red edges

then as no. of edges is $\frac{n(n-1)}{2}$ in a K_n, then X $\frac{n(n-1)}{4}$ not necessarily! number of blue & red edges.

for 6 i.e. $\frac{6(5)}{4} = \frac{30}{4} = 7.5$ but lets have 7 blue & 8 red. In the worst case

we could have one vertex with 5 blue vertexes X, then it must be connected to another (why is this worst?)

(7-5 = 2 blue vertices left)

vertex which has a blue vertex to another vertex, which is connected to the

we color edges, not vertices!

original vertex using a blue vertex. Hence we have a 3-cycle. This can be argued for

7 red X & 8 blue as well where not the only cases! a red 3-cycle is possible. As we increase the

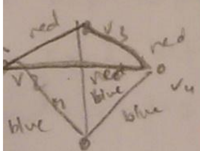
number of vertices, we increase the number of red or blue edges hence making it

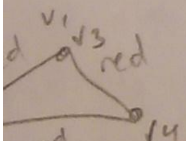
easier of have 3-cycles, and we also maintain the previous state of having a 3-cycle.

(b) In the previous part, you proved that at least three of the edges incident to v_1 are the same color. Without loss of generality, you may assume that color is blue. Suppose that $\{v_1, v_2\}$, $\{v_1, v_3\}$, and $\{v_1, v_4\}$ are blue edges. Prove that between these four vertices, there must exist either a blue 3-cycle or a red 3-cycle.

4 vertices. Total edges = $\frac{4(3)}{2} = 6$

red cycle





8

As we 4 vertices, we have 6 total edges hence ~~we have~~ we can have worst case 3 red edges. As all 3 blue edges are connected to v_1 to other vertices. The other vertices must be connected together using red edges only. Why? This is not a condition! as $\{v_2, v_4\}$, $\{v_3, v_4\}$ and $\{v_2, v_3\}$ are red edges, it is easily observable that these edges form a red 3-cycle. In the case we have 4 blue edges, we will have a random vertex connected to another random vertex by blue edge as these two are connected to v_1 by blue edges, we have a 3-cycle.

4. (18 points) Prove the combinatorial identity

$$\sum_{i=0}^{k} C(m+k-i-1,k-i)\cdot C(n+i-1,i) = C(m+n+k-1,k)$$

using a combinatorial argument. No more than half credit will be awarded to an algebraic proof. (Hint: Use Pirates and Gold.)

Proof: So we have m males, n females and k martians and we want to find k martian looking actors (humans look like martians) who will film in this new movie.

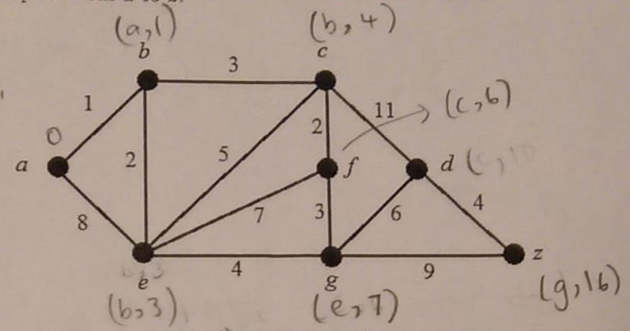
RHS: We gather all of them and remove one martian and count all the ways we could get k martians. This is just $C(m+n+k-1,k)$

LHS: we do the selection of k martians by partitioning by ~~how~~ the fact that the total number of men will be selected a certain number of times. let say the number of women get selected i times where $0 \le i \le k$, hence men then need to be selected k-i times. This can be represented by $C(m+k-i-1,k-i)\cdot C(n+i-1,i)$ where we add martians from one set to another depending on which sides gets selected more he the martians have the same chance of being selected.

We end up with $\sum_{i=0}^{k} C(m+k-i-1,k-i)\cdot C(n+i-1,i)$ to account for possibilities which is the same as RHS

5. (20 points) Run Dijkstra's algorithm on the following graph to find the shortest path from a to z. Recall that at each stage of Dijkstra's algorithm, one vertex is chosen and given a permanent label which represents the length of the shortest path from a to that vertex. ~~Write down~~ the list of vertices in the order in which they are given permanent labels. Additionally, find the length of a shortest path from a to z.

all vertices except "a" equate ∞



a) $L(a) = 0$

$T = \{b, c, d, e, f, g, z\}$

$L(b) = 0 + 1 = \min\{\infty, 1\} = 1$

$L(e) = 0 + 8 = 8$

b) $L(b) = 1$

$T = \{c, d, e, f, g, z\}$

$L(c) = 1 + 3 = \min\{\infty, 4\} = 4$

$L(e) = 1 + 2 = \min\{8, 3\} = 3$

c) $L(e) = 3$

$T = \{c, d, f, g, z\}$

$L(c) = 3 + 5 = \min\{4, 8\} = 4$

$L(f) = 3 + 7 = \min\{\infty, 10\} = 10$

$L(g) = 3 + 4 = \min\{\infty, 7\} = 7$

d) $L(c) = 4$

$T = \{d, f, g, z\}$

$L(f) = 4 + 2 = \min\{10, 6\} = 6$

$L(d) = 4 + 11 = \min\{\infty, 15\} = 15$

e) $L(f) = 6$

$T = \{d, g, z\}$

$L(g) = 6 + 3 = \min\{9, 7\} = 7$

f) $L(g) = 7$

$T = \{d, z\}$

$L(d) = 7 + 6 = \min\{13, 15\} = 13$

$L(z) = 7 + 9 = \min\{\infty, 16\} = 16$

g) $L(d) = 13$

$T = \{z\}$

$L(z) = 13 + 4 = \min\{16, 17\} = 16$

shortest path: $a \to b \to c \to g \to z$

length = 16

list of perm. label

Math 61, Lec 1
Winter 2016
Exam 2
2/22/16
Time Limit: 50 Minutes

Name (Print): Abdul Aziz Siddiqi

Name (Sign): Abdul Aziz

Discussion Section: 1E

This exam contains 6 pages (including this cover page) and 5 problems. Check to see if any pages are missing.

You may *not* use books, notes, or any calculator on this exam.

Unless otherwise stated in the problem, you may leave all answers in terms of $\binom{n}{k}$, $P(n,k)$, $k!$, or any sum, difference, product, or quotient of such symbols.

Partial credit will only be awarded to answers for which an explanation and/or work is shown.

Please attempt to organize your work in a reasonably neat and coherent way, in the space provided. If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score	
1	24	18	6
2	16	16	
3	22	10	12
4	18	18	
5	20	8	12
Total:	100	70	

1. (24 points) There are 999,999 natural numbers less than one million. We write any of them as a six digit number, including leading zeros. (For example, 001124 is how we write the number 1124).

(a) How many of these numbers have all different digits?

$= \cancel{9\times8\times7\times6\times5\times4}$

$=(10\times9\times8\times7\times6\times5)$

+8

(b) How many of these numbers have exactly four distinct digits? (For example, 922433 is valid, but 922435 is not valid and 922444 is not valid).

2211 3111

$= \underbrace{10\times9\times8\times7}_{\text{4 distinct}} \times \underbrace{P(4,2)}_{\text{permute 2 digits out of the 4 selected.}}$

$= \cancel{(10\times9\times8\times7)\times(4\times3)}$

+2 $\binom{10}{4}\left[\binom{4}{2}\frac{6!}{2!2!} + \binom{4}{1}\frac{6!}{3!}\right]$

(c) How many of these numbers have digits that sum to 18?

$x_1+x_2+x_3+x_4+x_5+x_6 = 18$

$0 \le x_1, x_2, x_3 \cdots x_6 \le 9$

without upper bond: $C(18+6-1, 6-1) = C(23,5)$

for $x_1, x_2, x_3 \cdots x_6 \ge 10$

$6\times C(8+6-1, 6-1) = 6C(13,5)$

so $= C(23,5) - 6C(13,5)$

+8

2. (16 points) Solve the recurrence relation $A_n = 3A_{n-1} + 4A_{n-2}$, where $A_0 = 3$, $A_1 = 7$.

$t^2 = 3t + 4$

$t^2 - 3t - 4 = 0$

$(t-4)(t+1) = 0$

$t = 4, \; t = -1$

$A_n = b(4)^n + c(-1)^n$

$A_0 = 3 = b + c$

$A_1 = 7 = 4b - c$ +

$10 = 5b$ $\quad c = 1$

$b = 2$

$A_n = 2(4)^n + (-1)^n$